Introduction to mathematical Statistics Quiz 2 Solution

Quiz 2, question 1. An exam consists of 10 multiple choice questions. Each question has 4 possible choices. Only 1 is correct. Jeff did not study for the exam. So he just guesses at the right answer for each question (pure guess, not an educated guess). What is his chance of passing the exam? That is, to make at least 6 correct answers.

Answer: Yes, this is a binomial experiment with n=10, p=0.25, "S"=choose the right answer for each question.

Let X be the total # of "S"

P(pass)=P($X \geq 6$)=P(X=6 or X=7 or X=8 or X=9 or X=10)

= P(X=6)+ P(X=7)+ P(X=8)+ P(X=9)+ P(X=10)

$$=\sum_{k=6}^{10}\binom{10}{k}0.25^k 0.75^{10-k} =$$

Quiz 2, question 2. Let X~B(n, p), please derive the m.g.f. of X. Please show the entire derivation for full credit.

m.g.f. of X

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} P(X = x)$$

$$= \sum_{x=0}^{n} e^{tx}\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x}$$

$$= [pe^t + (1-p)]^n$$

Theorem. $(a+b)^n = \sum_{x=0}^{n}\binom{n}{x}a^x b^{n-x}$

Quiz 2, question 3. Let $X_1 \sim B(n_1, p)$ and $X_2 \sim B(n_2, p)$. Furthermore, X_1 and X_2 are independent. Please derive the distribution of $Y = X_1 + X_2$

Solution. $M_Y(t) = E(e^{tY}) = E\left(e^{t(X_1+X_2)}\right) = E(e^{tX_1+tX_2}) = E(e^{tX_1} \cdot e^{tX_2}) = E(e^{tX_1}) \cdot E(e^{tX_2})$

$$= M_{X_1} \cdot M_{X_2} = [pe^t + (1-p)]^{n_1} \cdot [pe^t + (1-p)]^{n_2}$$

$= [pe^t + (1-p)]^{n_1+n_2}$

$$\therefore Y \sim B(n_1 + n_2, p)$$

Binomial Distribution (Review)

Def. Binomial Experiment:

1) It consists of n trials
2) Each trial results in 1 of 2 possible outcomes, "S" or "F"
3) The probability of getting a certain outcome, say "S", remains the same, from trial to trial, say P("S")=p
4) These trials are independent, that is the outcomes from the previous trials will not affect the outcomes of the up-coming trials

Def. Binomial Distribution:

Let X denotes the total # of "S" among the n trials then X~B(n, p)

Its probability density function (pdf), also referred to as the probability mass function (p.m.f.) since X is a discrete random variable is

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \; ; x = 0, 1, \cdots, n$$

In addition, we have:

$E(X) = np$

$Var(X) = np(1-p)$