Introduction to mathematical Statistics **Quiz 1 Solution**

1 (25 points). For a random variable following the normal distribution with mean μ and variance σ^2, please derive its moment generating function.

Solution:

Let X be a R.V, its Moment Generation Function (m.g.f.) is defined as a special mathematical expectation with g(X) = $E(e^{tX})$.

That is $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

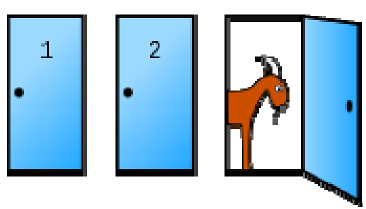
Now we derive the m.g.f. of $X \sim N(\mu, \sigma^2)$

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{tx - \frac{x^2-2\mu x+\mu^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2-2\mu x-2t\sigma^2 x+\mu^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+t\sigma^2))^2-(\mu+t\sigma^2)^2+\mu^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+t\sigma^2))^2}{2\sigma^2}} e^{\frac{2\mu t\sigma^2+t^2\sigma^4}{2\sigma^2}} dx = e^{\mu t+\frac{\sigma^2 t^2}{2}} \cdot 1 = e^{\mu t+\frac{\sigma^2 t^2}{2}}$$

2. (**The Monty Hall Problem; 25 points).** Suppose you are on a game show, and you are given a choice of three doors. Behind one door is a car, behind the others, goats. You would win if you have chosen the door with the car behind it. After you have chosen a door (say, door #1), the host, who knows exactly what are behind the three doors, would reveal one of the other two doors, and he would always choose one with a goat behind it, (say door #3). Then he would ask you, "Would you like to pick the other unopened door (door #2) instead – that is, to switch; or would you like to stay with the door you had chosen initially (door #1)?"
(a) What is the chance of winning if you do switch?
(b) What is the chance of winning if you do not switch?



http://en.wikipedia.org/wiki/Monty_Hall_problem

Solution:

(a) $P(Win\_By\_Switch)$

$$= P(WBSW \mid First\_Door\_Chosen\_Has\_Prize) \cdot P(F.D.C\_Has\_Prize)$$

$$+P(WBSW \mid First\_Door\_Chosen\_Has\_No\_Prize) \cdot P(F.D.C\_Has\_No\_Prize) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$$

(b) $P(Win\_By\_Stay)$

$$= P(WBST \mid First\_Door\_Chosen\_Has\_Prize) \cdot P(F.D.C\_Has\_Prize)$$

$$+P(WBST \mid First\_Door\_Chosen\_Has\_No\_\Pr ize)\cdot P(F.D.C\_Has\_No\_\Pr ize) \qquad =1\cdot\frac{1}{3}+0\cdot\frac{2}{3}=\frac{1}{3}$$

Note: Please note that the two probabilities should sum up to 1. More introductions to this problem can be found on line, for example, the wiki site: http://en.wikipedia.org/wiki/Monty_Hall_problem

3. Suppose the IQ of army recruits follow the normal distribution with mean 180 and variance 64.
(a) (12.5 points). For a randomly selected recruit, what is the probability that his IQ is higher than 190?
(b) (12.5 points). For 16 randomly selected recruits, what is the probability that their average IQ is higher than 190?

Solution:
(a) Let X denote the IQ of a randomly selected army recruit, then $X \sim N(180, 64)$

We want to calculate $P(X > 190)$

$$\begin{aligned}
&P(X > 190)\\
&= P(\frac{X-180}{8} > \frac{190-180}{8})\\
&= P(Z > 1.25)\\
&= 0.5 - 0.3944\\
&= 0.1056
\end{aligned}$$

So the chance is 10.56%.

(a) Let $\bar{X}$ denote the average IQ of 16 randomly selected army recruits, then $\bar{X} \sim N(180, \frac{64}{16}) = N(180, 4)$

We want to calculate $P(\bar{X} > 190)$

So the chance is 10.56%.

$$\begin{aligned}
&P(\bar{X} > 190)\\
&= P(\frac{\bar{X}-180}{2} > \frac{190-180}{2})\\
&= P(Z > 5)\\
&\approx 0
\end{aligned}$$

So the chance is near 0.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population $N\left(\mu, \sigma^2\right)$. Please

(a) (12.5 points) Derive the distribution of $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$

(b) (12.5 points) Derive the distribution of $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

Solution:
(a)

$$M_{\bar{X}}(t) = E(e^{t\bar{X}}) = E\left(e^{t\frac{\sum X_i}{n}}\right) = E\left(e^{\frac{t}{n}\sum X_i}\right)$$

$$= \prod_{i=1}^{n} E\left(e^{\frac{t}{n}X_i}\right) = \prod_{i=1}^{n}\left(e^{\mu\frac{t}{n}+\frac{\sigma^2}{2}\left(\frac{t}{n}\right)^2}\right) = e^{n\left(\mu\frac{t}{n}\right)+n\left(\frac{\sigma^2 t^2}{2n^2}\right)} = e^{\mu t+\frac{1}{2}\frac{\sigma^2}{n}t^2}$$

$$\Rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

(b)

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$

$$M_Z(t) = E(e^{tZ}) = E\left(e^{t\cdot\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}\right) = E\left(e^{\frac{t}{\sigma/\sqrt{n}}\cdot\bar{X}}\cdot e^{\frac{-\mu t}{\sigma/\sqrt{n}}}\right)$$

Let $t^* = \frac{t}{\sigma/\sqrt{n}}$

$$= e^{\frac{-\mu t}{\sigma/\sqrt{n}}} \times E(e^{t^*\bar{X}})$$

$$= e^{\frac{-\mu t}{\sigma/\sqrt{n}}} \times e^{\mu t^* + \frac{1}{2}\frac{\sigma^2}{n}(t^*)^2}$$

$$= e^{\frac{-\mu t}{\sigma/\sqrt{n}}} \times e^{\mu(\frac{t}{\sigma/\sqrt{n}})+\frac{1}{2}\frac{\sigma^2}{n}\left(\frac{t}{\sigma/\sqrt{n}}\right)^2}$$

$$= e^{0\cdot t+\frac{1}{2}t^2}$$

$$\rightarrow Z \sim N(0,1)$$

5. (extra credit for all). Let X and Y be random variables with joint pdf

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\},\ -\infty < x < \infty,$$

$-\infty < y < \infty$. Then X and Y are said to have the *bivariate normal distribution*. The joint moment generating function for X and Y is $M(t_1,t_2) = \exp\left[t_1\mu_X + t_2\mu_Y + \frac{1}{2}\left(t_1^2\sigma_X^2 + 2\rho t_1 t_2\sigma_X\sigma_Y + t_2^2\sigma_Y^2\right)\right]$.

(a) (8 points) Find the marginal pdf's of X and Y;

(b) (8 points) Prove that X and Y are independent if and only if ρ = 0. (Here ρ is indeed, the correlation coefficient between X and Y.)

(c) (9 points) Find the distribution of $(X+Y)$.

Solution:

(a)

$$M(t_1,0)=\exp\left(\mu_X t_1+\frac{1}{2}\sigma_X^2 t_1^2\right)\quad \therefore \text{X\~N}(\mu_X,\sigma_X^2)$$

$$M(0,t_2)=\exp\left(\mu_Y t_2+\frac{1}{2}\sigma_Y^2 t_2^2\right)\quad \therefore Y\text{\~N}(\mu_Y,\sigma_Y^2)$$

(b)

If $\rho=0$, then $M(t_1,t_2)=\exp\left[\mu_X t_1+\mu_Y t_2+\frac{1}{2}\left(\sigma_X^2 t_1^2+\sigma_Y^2 t_2^2\right)\right]=M(t_1,0)\cdot M(0,t_2)$

Therefore, X and Y are independent.

If X and Y are independent, then

$$M(t_1,t_2)=M(t_1,0)\cdot M(0,t_2)=\exp\left[\mu_X t_1+\mu_Y t_2+\frac{1}{2}\left(\sigma_X^2 t_1^2+\sigma_Y^2 t_2^2\right)\right]$$

$$=\exp\left[\mu_X t_1+\mu_Y t_2+\frac{1}{2}\left(\sigma_X^2 t_1^2+2\rho\sigma_X\sigma_Y t_1 t_2+\sigma_Y^2 t_2^2\right)\right]\text{ for any } X \text{ and } Y$$

Therefore, $\rho=0$

(c)

$$M_{X+Y}(t)=E\left[e^{t(X+Y)}\right]=E\left[e^{tX+tY}\right]$$

Recall that $M(t_1,t_2)=E\left[e^{t_1X+t_2Y}\right]$, therefore we can obtain $M_{X+Y}(t)$ by setting $t_1=t_2=t$ in $M(t_1,t_2)$

That is,

$$M_{X+Y}(t)=M(t,t)=\exp\left[\mu_X t+\mu_Y t+\frac{1}{2}\left(\sigma_X^2 t^2+2\rho\sigma_X\sigma_Y t^2+\sigma_Y^2 t^2\right)\right]$$

$$=\exp\left[\left(\mu_X+\mu_Y\right)t+\frac{1}{2}\left(\sigma_X^2+2\rho\sigma_X\sigma_Y+\sigma_Y^2\right)t^2\right]$$

$$\therefore X+Y\sim N(\mu=\mu_X+\mu_Y,\sigma^2=\sigma_X^2+2\rho\sigma_X\sigma_Y+\sigma_Y^2)$$

*****That's all, folks!*****