Introduction to mathematical Statistics Lecture8

Order Statistics & Their Applications in Point Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with p.d.f. f(x). Then,

$X_{(1)} = \min(X_1, X_2, \ldots, X_n)$

$X_{(n)} = \max(X_1, X_2, \ldots, X_n)$

and $X_{(1)} \le X_{(2)} \le \cdots \le X_{(k)} \le \cdots \le X_{(n)}$

p.d.f.'s for $X_{(1)}$ and $X_{(n)}$

W.L.O.G.(Without Loss of Generality), let's assume X is continuous.

$$P(X_{(1)} > x) = P(X_1 > x, X_2 > x, \ldots, X_n > x) = \prod_{i=1}^{n} P(X_i > x) = \prod_{i=1}^{n} [1 - F_{X_i}(x)]$$

$$F_{X_{(1)}}(x) = 1 - \prod_{i=1}^{n}[1 - F_{X_i}(x)]$$

$$f_{X_{(1)}}(x) = -\frac{d}{dx}\prod_{i=1}^{n}[1 - F_{X_i}(x)] = -\frac{d}{dx}\prod_{i=1}^{n}[1 - F(x)]^n = n[1 - F(x)]^{n-1}f(x)$$

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x) = \prod_{i=1}^{n} F_{X_i}(x) = [F(x)]^n$$

$$f_{X_{(n)}}(x) = n[F(x)]^{n-1}f(x)$$

Example. Let $X_n \overset{i.i.d}{\sim} \exp(\lambda)$, i=1,…,n

Please 1. Derive the MLE of λ

2. Derive the p.d.f. of $X_{(1)}$

3. Derive the p.d.f. of $X_{(n)}$

Solutions.

1.

$$L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n}\left(\lambda e^{-\lambda x_i}\right) = \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i}$$

$$l = \ln L = n\ln\lambda - \lambda\sum_{i=1}^{n} x_i$$

$$\frac{dl}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\lambda} = \frac{1}{\bar{X}}$$

Is $\hat{\lambda}$ an unbiased estimator of λ? $E\left(\frac{1}{\bar{X}}\right) = ?$

$$M_{X_i}(t) = \frac{\lambda}{\lambda - t}$$

$$M_{\sum_{i=1}^{n} x_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n$$

$$Y = \sum_{i=1}^{n} x_i \sim \text{gamma}(\lambda, n)$$

$$f_Y(y) = \frac{\lambda}{\Gamma(n)}(\lambda y)^{n-1} e^{-\lambda y}$$

$$\text{Let } Y = \sum_{i=1}^{n} x_i$$

$$E\left(\frac{1}{Y}\right) = \int_0^{\infty} \frac{1}{y} \frac{\lambda}{(n-1)!} (\lambda y)^{n-1} e^{-\lambda y} dy = \frac{\lambda}{n-1} \int_0^{\infty} \frac{\lambda}{(n-2)!} (\lambda y)^{n-2} e^{-\lambda y} dy = \frac{\lambda}{n-1}$$

$$E\left(\frac{1}{\bar{X}}\right) = n\left(\frac{\lambda}{n-1}\right) = \frac{n\lambda}{n-1} \neq \lambda$$

$\hat{\lambda}$ is not unbiased

2. $X_{(1)} = \min(X_1, X_2, \dots, X_n)$

$$P(X_{(1)} > x) = \prod_{i=1}^{n} P(X_i > x) = \prod_{i=1}^{n}[1 - F(x)] = [1 - F(x)]^n$$

$$F_{X_{(1)}}(x) = 1 - [1 - F(x)]^n$$

$$f_{X_{(1)}}(x) = n[1 - F(x)]^{n-1} f(x)$$

$$f(x) = \lambda e^{-\lambda x}, x > 0 \ (\textit{exponential distribution})$$

$$F(x) = \int_0^x f(u)du = \int_0^x \lambda e^{-\lambda u} du = [-e^{-\lambda u}]|_0^x = 1 - e^{-\lambda x}$$

$$f_{X_{(1)}}(x) = n\lambda e^{-\lambda x}[1 - (1 - e^{-\lambda x})]^{n-1} = n\lambda e^{-\lambda x}(e^{-\lambda x})^{n-1} = n\lambda(e^{-\lambda x})^n, x>0$$

3. $X_{(n)} = \max(X_1, X_2, \dots, X_n)$

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x) = \prod_{i=1}^{n} P(X_i \le x) = [F(x)]^n$$

$$f_{X_{(n)}}(x) = n[F(x)]^{n-1} f(x) = n[1 - e^{-\lambda x}]^{n-1} \lambda e^{-\lambda x}, x>0$$

Example. Let X be a random variable with pdf.

$$f(x) = \begin{cases} 1, \text{if } x \in [\theta - \frac{1}{2}, \theta + 1/2] \\ 0, \text{otherwise} \end{cases}$$

Derive the MLE of θ.

Solution.

Uniform Distribution $\Rightarrow$ important!!

$$L = \prod_{i=1}^{n} f(x_i) = \begin{cases} 1, \text{if all } x_i \in [\theta - \frac{1}{2}, \theta + 1/2] \\ 0, \text{otherwise} \end{cases}$$

MLE : max lnL -> max L

means => $\theta - \frac{1}{2} \le X_1 \le \theta + 1/2$

$\theta - \frac{1}{2} \le X_2 \le \theta + 1/2$

…

$\theta - \frac{1}{2} \le X_n \le \theta + 1/2$

Order statistics are useful to derive MLE.

$\theta - \frac{1}{2} \le X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)} \le \theta + 1/2$

$\theta \le X_{(1)} + \frac{1}{2}$

$\theta \ge X_{(n)} - \frac{1}{2}$

Therefore,

If $\theta \in \left[X_{(1)} + \frac{1}{2}, X_{(n)} - \frac{1}{2}\right]$, then $L = 1$

Therefore, any $\hat{\theta} \in \left[X_{(1)} + \frac{1}{2}, X_{(n)} - \frac{1}{2}\right]$ is an MLE for θ.

Mean Squared Error (M.S.E.)

How to evaluate an estimator?

Definition. Mean Squared Error

Let T=t($X_1, X_2, \ldots, X_n$) be an estimator of $\tau(\theta)$, then the M.S.E. of the estimator T is defined as :

$MSE_t(\tau(\theta)) = E\left[(T - \tau(\theta))^2\right]$: average squared distance from T to $\tau(\theta)$

$=E\left[(T - E(T) + E(T) - \tau(\theta))^2\right]$

$=E\left[(T - E(T))^2\right] + E\left[(E(T) - \tau(\theta))^2\right] + 2E[(T - E(T))(E(T) - \tau(\theta))]$

$=E\left[(T - E(T))^2\right] + E\left[(E(T) - \tau(\theta))^2\right] + 0$

$=Var(T) + (E(T) - \tau(\theta))^2$

Here $|E(T) - \tau(\theta)|$ is "the bias of T"

If unbiased, $(E(T) - \tau(\theta))^2 = 0$.

The estimator has smaller mean-squared error is better.

Example. Let $X_1, X_2, \ldots, X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$

M.L.E. for μ is $\hat{\mu} = \bar{X}$; M.L.E. for σ^2 is $\hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}$

1. M.S.E. of $\hat{\sigma}^2$?
2. M.S.E. of S^2 as an estimator of σ^2

Solution.

1. $MSE_{\hat{\sigma}^2}(\sigma^2) = E[(\hat{\sigma}^2 - \sigma^2)^2] = Var(\sigma^2) + (E(\hat{\sigma}^2) - \sigma^2)^2$

   To get $Var(\hat{\sigma}^2)$, there are 2 approaches.

   a. By the first definition of the Chi-square distribution.

Note $X_i \sim N(\mu, \sigma^2),\ W = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi_{n-1}^2$, Gamma$(\lambda = \frac{1}{2}, s = \frac{n-1}{2})$

$E(W) = \frac{s}{\lambda} = n-1,\ Var(W) = \frac{s}{\lambda^2} = 2(n-1)$

$Var(\hat{\sigma}^2) = Var(\frac{W}{n}\sigma^2) = \frac{\sigma^4}{n^2}Var(W) = \frac{\sigma^4}{n^2}2(n-1)$

b. By the second definition of the Chi-squre distribution.

For Z~N(0,1), W=$\sum_{i=1}^{n} Z_i^2$

$$Var(Z^2) = E\left[\left(Z^2 - E(Z^2)\right)^2\right] = E\left[\left(Z^2 - \left(var(Z^2) + E(Z)\right)\right)^2\right] = E[(Z^2-1)^2]$$

$$\text{Since } Var(Z) = E(Z^2) - E(Z) = 1 \text{ from } Z \sim N(0,1), E(Z^2) = 1 = E[Z^4 - 2E(Z^2) + 1]$$
$$= E(Z^4) - 1$$

Calculate the 4th moment of Z~N(0,1) using the mgf of Z;

$M_Z(t) = e^{t^2/2}$

$M'_Z(t) = te^{t^2/2}$

$M''_Z(t) = te^{t^2/2} + t^2e^{t^2/2}$

$M^{(3)}_Z(t) = 3te^{t^2/2} + t^2e^{t^2/2}$

$M^{(4)}_Z(t) = 3e^{t^2/2} + 6t^2e^{t^2/2} + t^4e^{t^2/2}$

Set $t = 0$, $M^{(4)}_Z(0) = 3 = E(Z^4)$

$Var(Z^2) = 3 - 1 = 2$

$$Var(W) = \sum_{i=1}^{n-1} Var(Z_i^2) = 2(n-1)$$

$$\hat{\sigma}^2 = \frac{\sigma^2}{n}W,\ Var(\hat{\sigma}^2) = \frac{\sigma^4}{n^2}2(n-1)$$

$$MSE_{\hat{\sigma}^2}(\sigma^2) = Var(\hat{\sigma}^2) + [E(\hat{\sigma}^2) - \sigma^2]^2 = \frac{2(n-1)}{n^2}\sigma^4 + [E\left(\frac{n-1}{n}S^2\right) - \sigma^2]^2$$

$$= \frac{2(n-1)}{n^2}\sigma^4 + \left[\frac{n-1}{n}\sigma^2 - \sigma^2\right]^2 \text{ (we know } E(S^2) = \sigma^2) = \frac{2n-1}{n^2}\sigma^4$$

The M.S.E. of $\hat{\sigma}^2$ is $\frac{2n-1}{n^2}\sigma^4$

We know S^2 is an unbiased estimator of σ^2

$$E[(S^2-\sigma^2)^2] = Var(S^2) + 0 = Var\left(\frac{\sigma^2 W}{n-1}\right) = \left(\frac{\sigma^2}{n-1}\right)^2 var(W) = \frac{2\sigma^4}{n-1}$$

Please read Cramer-Rao Lower Bound (5.5.2) before our next lecture.
Chapter 5 parts that will not be covered in our lectures: 5.4.2 (Sufficiency), 5.5.1 (Consistency).