Introduction to mathematical Statistics Lecture 5

Binomial Distribution

Eg. 1. Suppose each child's birth will result in either a boy or a girl with equal probability. For a randomly selected family with 2 children, what is the chance that the chosen family has 1) 2 boys? 2) 2 girls? 3) a boy and a girl?

Solution: 25%; 25%; 50%

P(B and B)= P(B∩B)= P(B)·P(B)=0.5·0.5=0.25

P(G and G)= P(G∩G)= P(G)·P(G)=0.5·0.5=0.25

P(B and G)= $P(B_1 \cap G_2$ or $B_2 \cap G_1) = P((B_1 \cap G_2) \cup (B_2 \cap G_1))$

Def. Binomial Experiment:

1) It consists of n trials
2) Each trial results in 1 of 2 possible outcomes, "S" or "F"
3) The probability of getting a certain outcome, say "S", remains the same, from trial to trial, say P("S")=p
4) These trials are independent, that is the outcomes from the previous trials will not affect the outcomes of the up-coming trials

Eg. 1 (continued) n=2, let "S"=B, P(B)=0.5

Let X denotes the total # of "S" among the n trials then X~B(n, p)

Its probability density function (pdf) is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x=0,1,\ldots,n$$

**For a discrete random variable, its pdf is also called its probability mass function (pmf)

$$\sum_{i=0}^{n} P(X = x) = 1$$

Eg. 1 (continued) n=2, p=0.5, S=B, birth=trial

Answer: Let X denotes the total of boys form the 2 births. Then X~B(n=2, p=0.5)

1) P(2 boys)=P(X=2)= $\binom{2}{2} 0.5^2 (1-0.5)^{2-2}$ =.25

2) P(2 girls)=P(X=0)= $\binom{2}{0}0.5^0(1-0.5)^{2-0}$ =.25

3) P(1 boys and a girl)=P(X=1)= $\binom{2}{1}0.5^1(1-0.5)^{2-1}$ =.5

4) What is the probability of having at least 1 boy?

   P(at least 1 boy)=P(X$\geq$1)=P(X=1)+P(X=2) =.5+.25=.75

Eg 2. An exam consists of 10 multiple choice questions. Each question has 4 possible choices. Only 1 is correct. Jeff did not study for the exam. So he just guesses at the right answer for each question (pure guess, not an educated guess). What is his chance of passing the exam? That is, to make at least 6 correct answers.

Answer: Yes, this is a binomial experiment with n=10, p=0.25, "S"=choose the right answer for each question.

Let X be the total # of "S"

P(pass)=P(X$\geq$6)=P(X=6 or X=7 or X=8 or X=9 or X=10)

= P(X=6)+ P(X=7)+ P(X=8)+ P(X=9)+ P(X=10)

$$=\sum_{k=6}^{10}\binom{10}{k}0.25^k 0.75^{10-k}$$

** Please finish this exercise on your own. It is question 1 of Quiz 2.

Review: Mathematical Expectation.

Continuous random variable: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$

Discrete random variable: $E[g(X)] = \sum_{all\ x} g(x) f(x)$

Special case:

1) **(population) Mean**: $\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$

2) **(population) Variance**:

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty}(x-\mu)^2 \cdot f(x)dx = E(X^2)-[E(X)]^2$$

** $Var(X) = E[(X-\mu)^2] = E(X^2 - 2\mu X + \mu^2)$

$$= EX^2 - 2\mu EX + \mu^2 = EX^2 - \mu^2 = EX^2 - (EX)^2$$

3) Moment generating function:

Suppose X is a continuous random variable (R.V.) with probability density function (pdf) f(x). The moment generating function (mgf) of X is defined as

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx, \text{ where t is a small positive variable.}$$

$$E(X) = M_x^{'}(t=0)$$

$$E(X^2) = M_x^{''}(t=0)$$

For normal distribution, $X \sim N(\mu, \sigma^2)$, $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \; -\infty < x < \infty$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Theorem. If X_1, X_2 are independent, then

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$

Theorem. If X_1, X_2 are independent, then

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$$

Proof. X_1, X_2 are independent iff (if and only if) $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$, thus

$$M_{X_1+X_2}(t) = E(e^{t(X_1+X_2)}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(x_1+x_2)} f_{X_1,X_2}(x_1,x_2)dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} e^{tx_1} f_{X_1}(x_1)dx_1 \int_{-\infty}^{\infty} e^{tx_2} f_{X_2}(x_2)dx_2 = M_{X_1}(t) M_{X_2}(t)$$

Theorem. Under regularity conditions, there is 1-1 correspondence between the pdf and the mgf of a given random variable X. That is,

$$pdf\ f(x) \xleftrightarrow{1-1} mgf\ M_X(t).$$

Note: One could use this property to identify the probability distribution based on the moment generating function.

Quiz #2. (***Take home quiz. *** Due Monday, February 21, 2011, before class.)

1. An exam consists of 10 multiple choice questions. Each question has 4 possible choices. Only 1 is correct. Jeff did not study for the exam. So he just guesses at the right answer for each question (pure guess, not an educated guess). What is his chance of passing the exam? That is, to make at least 6 correct answers.

2. Let X~B(n, p), please derive the m.g.f. of X. Please show the entire derivation for full credit.

3. Let X_1, X_2 be two independent random variables following binomial distributions $B(n_1,p)$ and $B(n_2,p)$ respectively. Please derive the distribution of X_1+X_2. Please show the entire derivation for full credit.