Introduction to mathematical Statistics **Lecture 3**

***1. Review: The derivative and integral of some important functions one should remember.**

$$\frac{d}{dx}(x^k) = kx^{k-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\int_a^b x^k dx = (\frac{1}{k+1}x^{k+1})\Big|_{x=a}^{x=b} = \frac{1}{k+1}(b^{k+1} - a^{k+1})$$

$$\int_a^b e^x dx = e^x \Big|_{x=a}^{x=b} = e^b - e^a$$

*The Chain Rule

$$\frac{d}{dx} g[f(x)] = g'[f(x)] \cdot f'(x)$$

For example: $\frac{d}{dx} e^{x^2} = e^{x^2} \cdot 2x$

*The Product Rule

$$\frac{d}{dx}[g(x) \cdot f(x)] = g'(x) f(x) + g(x) f'(x)$$

***2. MGF, its second function: The m.g.f. will also generate the moments**

Moment: 1st (population) moment $E(X) = \int x \cdot f(x) dx$

2nd (population) moment $E(X^2) = \int x^2 \cdot f(x) dx$

…

Kth (population) moment $E(X^k) = \int x^k \cdot f(x) dx$

Take the Kth derivative of the $M_X(t)$ at t=0. , we get the Kth moment of X

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = E(X)$$

$$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = E(X^2)$$

...

$$\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = E(X^k)$$

Example: When $X \sim N(\mu,\sigma^2)$, we want to verify the above equations for k=1 & k=2.

$$\frac{d}{dt}M_X(t)=(e^{\mu t+\frac{1}{2}\sigma^2t^2})\cdot(\mu+\sigma^2 t)$$ (using the chain rule)

So when t=0

$$\frac{d}{dt}M_X(t)\Big|_{t=0}=\mu=E(X)$$

$$\frac{d^2}{dt^2}M_X(t)$$

$$=\frac{d}{dt}[\frac{d}{dt}M_X(t)]$$

(using the result of the Product Rule)

$$=\frac{d}{dt}[(e^{\mu t+\frac{1}{2}\sigma^2t^2})\cdot(\mu+\sigma^2 t)]$$

$$=(e^{\mu t+\frac{1}{2}\sigma^2t^2})\cdot(\mu+\sigma^2 t)^2+(e^{\mu t+\frac{1}{2}\sigma^2t^2})\cdot\sigma^2$$

And

$$\frac{d^2}{dt^2}M_X(t)\Big|_{t=0}$$

$$=\mu^2+\sigma^2$$ Considering $\sigma^2=Var(X)=E(X^2)-\mu^2$

$$=E(X^2)$$

*3. Review: Bivariate Normal Random Variable

$(X,Y)\sim BN(\mu_x,\sigma_x^2;\mu_y,\sigma_y^2;\rho)$ where ρ is the correlation between X & Y

The joint p.d.f. of (X,Y) is

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2\rho xy}{\sigma_x\sigma_y}\right)\right)$$

4. Homework 2 (Solution)

Q1. $Let\ X_1\ \&\ X_2\ be\ independent\ N(\mu,\sigma^2)$,

$prove\ that\ M_{X_1+X_2,X_1-X_2}(t_1,t_2)=M_{X_1+X_2}(t_1)M_{X_1-X_2}(t_2)$

Solution:

$Let\ W = X_1 + X_2\ and\ Y = X_1 - X_2$

$W\ and\ Y\ are\ independent\ if\ and\ only\ if\ \ M_{W,Y}(t_1, t_2) = M_W(t_1)M_Y(t_2)$

$$M_{W,Y}(t_1, t_2) = E\left(e^{t_1 W + t_2 Y}\right) \xrightarrow{Substitute\ for\ W\ \&\ Y} E\left(e^{t_1(X_1+X_2)+t_2(X_1-X_2)}\right) = E\left(e^{(t_1+t_2)X_1+(t_1-t_2)X_2}\right)$$

$$E\left(e^{(t_1+t_2)X_1+(t_1-t_2)X_2}\right) = E\left(e^{(t_1+t_2)X_1}\right)E\left(e^{(t_1-t_2)X_2}\right)\ \ because\ X_1\ \&\ X_2\ are\ independent$$

$$\cdot\ E\left(e^{(t_1+t_2)X_1}\right) = e^{\mu(t_1+t_2)+\frac{1}{2}(t_1+t_2)^2\sigma^2} = e^{\mu t_1+\mu t_2+\frac{\sigma^2 t_1^2}{2}+\sigma^2 t_1 t_2+\frac{\sigma^2 t_2^2}{2}}$$

$$\cdot\ E\left(e^{(t_1-t_2)X_2}\right) = e^{\mu(t_1-t_2)+\frac{1}{2}(t_1-t_2)^2\sigma^2} = e^{\mu t_1-\mu t_2+\frac{\sigma^2 t_1^2}{2}-\sigma^2 t_1 t_2+\frac{\sigma^2 t_2^2}{2}}$$

$$\therefore E\left(e^{(t_1+t_2)X_1+(t_1-t_2)X_2}\right) = e^{\mu t_1+\mu t_2+\frac{\sigma^2 t_1^2}{2}+\sigma^2 t_1 t_2+\frac{\sigma^2 t_2^2}{2}} e^{\mu t_1-\mu t_2+\frac{\sigma^2 t_1^2}{2}-\sigma^2 t_1 t_2+\frac{\sigma^2 t_2^2}{2}}$$

$$= \exp\left(\mu t_1 + \mu t_2 + \frac{\sigma^2 t_1^2}{2} + \sigma^2 t_1 t_2 + \frac{\sigma^2 t_2^2}{2} + \mu t_1 - \mu t_2 + \frac{\sigma^2 t_1^2}{2} - \sigma^2 t_1 t_2 + \frac{\sigma^2 t_2^2}{2}\right)$$

$$= \exp\left(2\mu t_1 + \frac{1}{2}(2\sigma^2)t_1^2 + \frac{1}{2}(2\sigma^2)t_2^2\right)$$

$$= e^{2\mu t_1+\frac{1}{2}(2\sigma^2)t_1^2} e^{0 t_2+\frac{1}{2}(2\sigma^2)t_2^2} = M_{X_1+X_2}(t_1)M_{X_1-X_2}(t_2)$$

$$- M_W(t_1)M_Y(t_2)$$

$$\therefore M_{W,Y}(t_1, t_2) = M_W(t_1)M_Y(t_2)$$

$\therefore W = X_1 + X_2$ *and* $Y = X_1 - X_2$ are independent

Q2. *Let* Z *be* $N(0,1)$, (1) please derive the covariance between Z *and* Z^2; (2) Are Z *and* Z^2

independent?

Solution:

(1) $cov(Z,\ Z^2) = E[(Z - E(Z))(Z^2 - E(Z^2))]$

$= E[(Z-0)(Z^2-1)] = E[Z^3] = \frac{\partial^3}{\partial t^3}[exp(\frac{1}{2}t^2)]|_{t=0} = 1$

(2) $Z\ and\ Z^2$ are not independent. You can do it in many ways, using (a) pdf, (b) cdf/probabilities, or (c) mgf.

For example, $P(Z>1,\ Z^2>1) = P(Z>1) \neq P(Z>1) * P(Z^2>1) = [P(Z>1)]^2$

Q3. Let $X \sim N(3,4)$, please calculate $P(1 < X < 3)$

Solution:

$$
\begin{aligned}
&P(1 < X < 3) \\
&= P\left(\frac{1-3}{2} < \frac{X-3}{2} < \frac{3-3}{2}\right) \\
&= P(-1 < Z < 0) \\
&= P(0 < Z < 1) \\
&= 0.3413
\end{aligned}
$$

Note: In the above, $Z \sim N(0,1)$