Introduction to mathematical Statistics Lecture2

1. Review of Normal Distribution (continued)

$X \sim N(\mu, \sigma^2)$

$\mu : Mean,\ \sigma^2 : Variance$

4) The Z-score transformation of any normal random variable X

Question: Let $X \sim N(\mu, \sigma^2)$**, please show that** $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Approach A: Prove through the pdf, that is, show $f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}$

Approach B: Prove through the m.g.f. (Moment Generation Function)
There is 1-1 correspondence between the pdf, and the cdf, and the mgf. Therefore if we can recognize the format of the mgf of a random variable, we will be able to declare that the given R.V. follows a certain probability distribution.

Now we derive the m.g.f. of Z:

$$M_Z(t) = E(e^{Zt}) = E(e^{t\cdot\frac{X-\mu}{\sigma}}) = E(e^{t\frac{X}{\sigma}} \cdot e^{-t\frac{\mu}{\sigma}}) = e^{-t\frac{\mu}{\sigma}} E(e^{\frac{t}{\sigma}X})$$

$$let\ t^* = \frac{t}{\sigma}$$

$$M_Z(t) = e^{-t\frac{\mu}{\sigma}} E(e^{t^* X}) = e^{-t\frac{\mu}{\sigma}} e^{\mu t^* + \frac{\sigma^2 t^{*2}}{2}} = e^{-t\frac{\mu}{\sigma} + \mu\frac{t}{\sigma} + \frac{\sigma^2 \left(\frac{t}{\sigma^2}\right)^2}{2}} = e^{\frac{t^2}{2}}$$

$$\therefore Z \sim N(0,1)$$

** Given the Z-score transformation, we are now able to calculate the probabilities for any normal random variable X as follows:

$$P(a \le X \le b)$$
$$= P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$$
$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$
$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

Example: $X \sim N(\mu, \sigma^2), \mu = 3, \sigma^2 = 10, a = 5, b = 7$

2. Joint distribution, and independence

Definition. The joint moment generating function of two random variables X and Y is defined as $M_{X,Y}(t_1, t_2) = E(e^{t_1 X + t_2 Y})$

Theorem. Two random variables X and Y are independent ⇔ (if and only if)
$f_{X,Y}(x,y) = f_X(x)f_Y(y) \Leftrightarrow M_{X,Y}(t_1,t_2) = M_X(t_1)M_Y(t_2)$

Definition. The covariance of two random variables X and Y is defined as
$COV(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$.

Theorem. If two random variables X and Y are independent, then we have $COV(X,Y) = 0$
However, $COV(X,Y) = 0$ does not necessarily mean that X and Y are independent

<u>Homework Assignment #2 (Due Monday, February 7, before class)</u>

Q1. $Let\ X_1\ \&\ X_2\ be\ independent\ N(\mu, \sigma^2)$,

$prove\ that\ M_{X_1+X_2, X_1-X_2}(t_1, t_2) = M_{X_1+X_2}(t_1)M_{X_1-X_2}(t_2)$

Q2. $Let\ Z\ be\ N(0,1)$, (1) please derive the covariance between $Z\ and\ Z^2$; (2) Are $Z\ and\ Z^2$ independent?

Q3. Let $X \sim N(3,4)$, please calculate $P(1 < X < 3)$