Introduction to mathematical Statistics

The Likelihood Ratio Test

(Another approach to construct a test)

Example 1. Let $X_1, X_2, \cdots X_n$ be a random sample from $N(\mu, \sigma^2)$ where σ^2 is known. We wish to test $H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$ at the significance level α. Please derive the test using the likelihood ratio method.

Solution.

1. ***Write down your parameter space under*** H_0

   $\omega = \{\mu : \mu = \mu_0\}$

2. ***Write down the unrestricted/original parameter space.***

   $\Omega = \{\mu : \mu \in R\}$

3. ***Write down the likelihood (of the data)***

$$L = f(x_1, x_2, \cdots, x_n; \mu)$$

$$= \prod_{i=1}^{n} f(x_i; \mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$$

4. ***Write down your log-likelihood.***

$$l = \ln L = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

5. ***Find your MLE under*** ω ***and plug it in to your*** L ***to obtain*** $\max_{\omega} L$

$$\max_{\omega} L = L(x_1, x_2, \cdots, x_n; \mu_0) = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma^2}}$$

6. ***Find the MLE(s) under*** Ω ***and plug in to your*** L ***to obtain*** $\max_{\Omega} L$

$$\frac{d\ln L}{d\mu}=0 \Rightarrow \frac{2\sum_{i=1}^{n}(x_i-\mu)}{2\sigma^2}=0 \Rightarrow \hat{\mu}=\overline{X}$$

$$\max_{\Omega} L = L(x_1,x_2,\cdots,x_n;\hat{\mu}) = (2\pi\sigma^2)^{-\frac{n}{2}}\cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2\sigma^2}}$$

7. Get the likelihood ratio

$$LR=\frac{\max_{\omega} L}{\max_{\Omega} L}=\frac{(2\pi\sigma^2)^{-\frac{n}{2}}\cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{-\frac{n}{2}}\cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2\sigma^2}}}=e^{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2-\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma^2}} \text{ , } 0\prec LR\le 1 \ (\because \omega\subseteq\Omega)$$

8. Derive the decision rule for a given significance level α

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$$
$$= P(LR\le c \mid H_0:\mu=\mu_0)$$

$$= P\left(e^{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2-\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma^2}}\le c \mid H_0{:}\mu{=}\mu_0\right)$$

$$= P\left(\sum_{i=1}^{n}(x_i-\bar{x})^2-\sum_{i=1}^{n}(x_i-\mu_0)^2\le 2\sigma^2\ln c \mid H_0{:}\mu{=}\mu_0\right)$$

Recall the z-test we derived before using the Pivotal Quantity method.

Test Statistic : $Z_0=\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}\overset{H_0}{\sim}N(0,1)$

$H_0:\mu=\mu_0$ vs. $H_a:\mu\ne\mu_0$

At α *, we reject* H_0 *if* $|Z_0|\ge Z_{\alpha/2}$

$$= P\left(\sum_{i=1}^{n}(x_i^2-2\bar{x}x_i+\bar{x}^2)-\sum_{i=1}^{n}(x_i^2-2x_i\mu_0+\mu_0^2)\le 2\sigma^2\ln c \mid H_0\right)$$

$$= P(-2n\bar{x}^2+n\bar{x}^2+2n\bar{x}\mu_0-n\mu_0^2\le 2\sigma^2\ln c \mid H_0)$$

$$= P(-n\bar{x}^2+2n\bar{x}\mu_0-n\mu_0^2\le 2\sigma^2\ln c \mid H_0)$$

$$= P\left(\bar{x}^2-2\bar{x}\mu_0+\mu_0^2\ge\frac{-2\sigma^2\ln c}{n} \mid H_0\right)$$

$$= P((\bar{x}-\mu_0)^2 \ge \frac{-2\sigma^2 \ln c}{n} \mid H_0{:}\mu{=}\mu_0)$$

$$= P((\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}})^2 \ge \frac{\frac{-2\sigma^2 \ln c}{n}}{(\sigma/\sqrt{n})^2} \mid H_0{:}\mu{=}\mu_0)$$

$$= P(Z_0^2 \ge c^* \mid H_0{:}\mu{=}\mu_0),\ c^* = -2\ln c\ (c = e^{-\frac{c^*}{2}})$$

$$= P(|Z_0| \ge \sqrt{c^*} \mid H_0 : \mu = \mu_0)$$

$$\therefore \sqrt{c^*} = Z_{\alpha/2}$$

$\therefore$ At α, we reject H_0 if $|Z_0| \ge Z_{\alpha/2}$

Example 2. Let $X_1, X_2, \cdots X_n$ be a random sample from $N(\mu,\sigma^2)$ where σ^2 is unknown. Please derive a test for $H_0 : \mu = \mu_0$ versus $H_a : \mu \ne \mu_0$ at the significance level α.

Solution.

1. $\omega = \{(\mu,\sigma^2) : \mu = \mu_0, \sigma^2 \succ 0\}$

2. $\Omega = \{(\mu,\sigma^2) : \mu \in R, \sigma^2 \succ 0\}$

3. $L = f(x_1, x_2, \cdots, x_n; \mu, \sigma^2)$

$$= \prod_{i=1}^{n} f(x_i;\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$$

4. $l = \ln L = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$

5. Find MLEs under ω and plug in to get $\max_{\omega} L$

$$\frac{dl}{d\sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\sigma^4} = 0$$

$$\Rightarrow \hat{\sigma}_\omega^2 = \frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{n}$$

$$\max_{\omega} L = L(x_1, x_2, \cdots, x_n; \mu_0, \hat{\sigma}_\omega^2)$$

$$= \left(2\pi \frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{n} \right)^{-\frac{n}{2}} \cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{n}}}$$

$$= (2\pi)^{-\frac{n}{2}} \left(\frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{n} \right)^{-\frac{n}{2}} e^{-\frac{n}{2}}$$

6. Find MLEs under Ω and plug in to get $\max_{\Omega} L$

$$\begin{cases} \frac{\partial l}{\partial \mu} = \frac{\sum_{i=1}^{n}(x_i - \mu)}{\sigma^2} = 0 \\ \frac{\partial l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^4} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{\mu}_\Omega = \overline{X} \\ \hat{\sigma}_\Omega^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n} \end{cases}$$

$$\max_{\Omega} L = L(x_1, x_2, \cdots, x_n; \hat{\mu}_\Omega, \hat{\sigma}_\Omega^2)$$

$$= \left(2\pi \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n} \right)^{-\frac{n}{2}} \cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{2\frac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{n}}}$$

$$=(2\pi)^{-\frac{n}{2}}\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}\right)^{-\frac{n}{2}}\cdot e^{-\frac{n}{2}}$$

7. $$LR=\frac{\max_{\omega} L}{\max_{\Omega} L}=\left(\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{-\frac{n}{2}}$$

8. Derive the decision rule based on significance level α

$$\alpha=P(\text{Reject } H_0 \mid H_0 \text{ is true})$$

$$=P(LR\le c \mid H_0:\mu=\mu_0)$$

$$=P(\left(\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{-\frac{n}{2}}\le c \mid H_0{:}\mu{=}\mu_0)$$

Recall the t-test statistic : $T_0=\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\overset{H_0}{\sim}t_{n-1}$

At α *, we reject* H_0 *in favor of* H_a *if* $|T_0|\ge t_{n-1,\alpha/2}$

$$=P(\left(\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{\frac{n}{2}}\ge\frac{1}{c} \mid H_0{:}\mu{=}\mu_0)$$

$$=P(\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\ge\left(\frac{1}{c}\right)^{\frac{2}{n}} \mid H_0{:}\mu{=}\mu_0)$$

$$=P(\frac{\sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\ge c^{*} \mid H_0{:}\mu{=}\mu_0)$$

$$= P\left(\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + 2(\bar{x} - \mu_0)\sum_{i=1}^{n}(x_i - \bar{x}) + n(\bar{x} - \mu_0)^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \geq c^* \mid H_0{:}\mu{=}\mu_0\right)$$

$$= P\left(1 + \frac{n(\bar{x} - \mu_0)^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \geq c^* \mid H_0{:}\mu{=}\mu_0\right)$$

$$= P(T_0^2 \geq c^{**} \mid H_0{:}\mu{=}\mu_0)$$

$$= P(|T_0| \geq \sqrt{c^{**}} \mid H_0{:}\mu{=}\mu_0)$$

$\therefore$ At α, we reject H_0 if $|T_0| \geq t_{n-1,\alpha/2}$

$\therefore$ The LR test is equivalent to the t-test.

Example 3. If $X_1, X_2, \cdots, X_n$ are a random sample from the exponential distribution with parameter λ. You can derive the LR test for $H_0 : \lambda = \lambda_0$ vs $H_a : \lambda \neq \lambda_0$.

Example 4. If $X_1, X_2, \cdots, X_n$ are a random sample from the $Poisson(\theta)$. You can derive the LR test for $H_0 : \theta = \theta_0$ vs $H_a : \theta \neq \theta_0$.