Introduction to mathematical Statistics

Chapter 7

Hypothesis Testing

Example.

H_0 (***null hypothesis***) : the average height is 5'8" or more. $(H_0 : \mu \geq 5'8")$

H_a (***alternative hypothesis***) : the average height is less than 5'8" $(H_a : \mu \prec 5'8")$

Example. (law suit) O.J. Simpson case

H_0 : O.J. is innocent. (***original belief***)

H_a : O.J. is guilty.

Truth

Decision		H_0	H_a
	H_0	Correct Decision	***Type II error***
	H_a	***Type I error***	Correct Decision

$P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0) = \alpha$

This probability is called the ***significance level of the test***.

Scenario 1 : Inference on one population mean μ (normal population, σ^2 is known)

There are two methods.

1. ***Pivotal Quantity Method*** (We will learn this method first.)
2. Likelihood Ratio Test (We will learn this method later.)

Now we introduce the general procedures in the Pivotal Quantity method.

1. Before deriving the test, you must know your Pivotal Quantity for the given problem:

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$$

2. Now you can start your hypothesis test. First write down your hypotheses. There are 3 cases (or choices). You need to choose the one that is most suitable for your given problem.

   a. (one-sided test) $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \succ \mu_0 \end{cases}$ or equivalently, $\begin{cases} H_0 : \mu \le \mu_0 \\ H_a : \mu \succ \mu_0 \end{cases}$

   b. (one-sided test) $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \prec \mu_0 \end{cases}$ or equivalently, $\begin{cases} H_0 : \mu \ge \mu_0 \\ H_a : \mu \prec \mu_0 \end{cases}$

   c. (two-sided test) $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \ne \mu_0 \end{cases}$

In the following we introduce the remainder of the test derivation procedures for each pair of hypotheses.

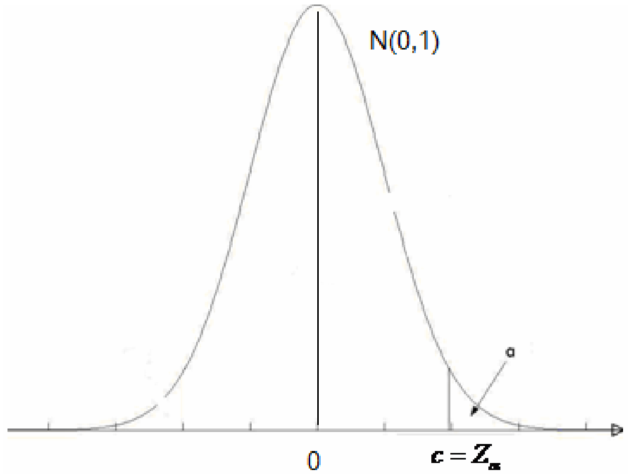
Case A. Suppose that we are testing $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \succ \mu_0 \end{cases}$

3. Now you write down your **Test Statistic** (which is your pivotal quantity with the value of the parameter under H_0 plugged in. Here we plug in $\mu = \mu_0$):

Test statistic: $Z_0 = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} \overset{H_0:\mu=\mu_0}{\sim} N(0,1)$

4. Now, we derive the decision rule based on α. (α is the significance level of the test.)

$\alpha = P(\text{Reject } H_0 \mid H_0) = P(Z_0 \ge c \mid H_0 : \mu = \mu_0)$



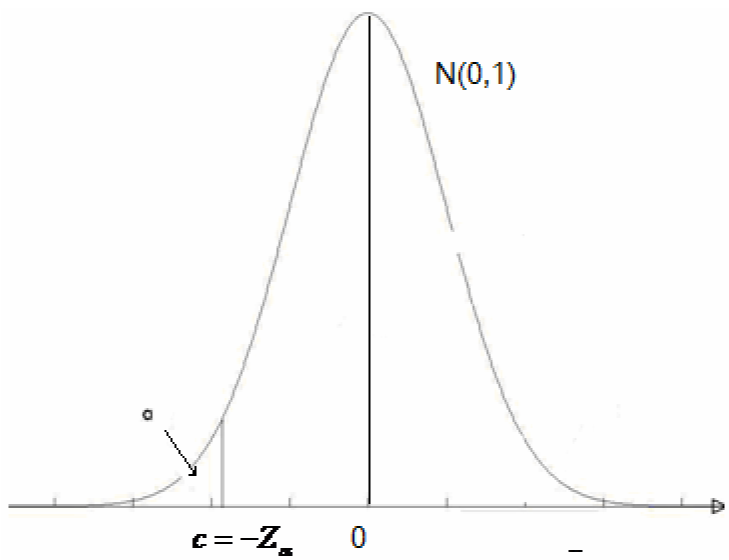
Thus we will reject H_0 in favor of H_a if $Z_0 \geq Z_\alpha$.

For example, $\alpha = 0.05 \Rightarrow$ Reject H_0 if $Z_0 \geq Z_{0.05} = 1.645$

Case B, now suppose we are testing $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \prec \mu_0 \end{cases}$,

Test Statistic : $Z_0 = \dfrac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} \overset{H_0:\mu=\mu_0}{\sim} N(0,1)$

$\alpha = P(\text{Reject } H_0 \mid H_0) = P(Z_0 \leq c \mid H_0 : \mu = \mu_0)$



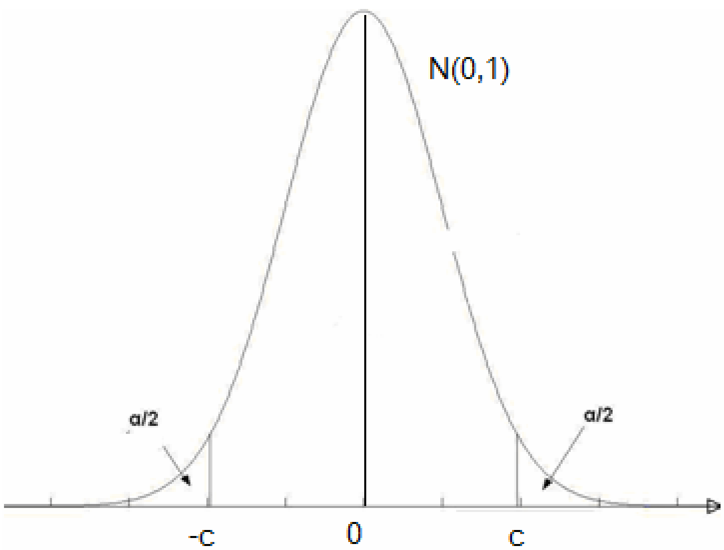
We will reject H_0 in favor of H_a if $Z_0 \le -Z_\alpha$.

Case C, now suppose we are testing $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \ne \mu_0 \end{cases}$,

Test Statistic : $Z_0 = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \overset{H_0:\mu=\mu_0}{\sim} N(0,1)$

$$\begin{aligned}\alpha &= P(\text{Reject } H_0 \mid H_0) \\ &= P(|Z_0| \ge c \mid H_0 : \mu = \mu_0) \\ &= P(Z_0 \ge c \mid H_0) + P(Z_0 \le -c \mid H_0) \\ &= 2 \times P(Z_0 \ge c \mid H_0)\end{aligned}$$

or $\dfrac{\alpha}{2} = P(Z_0 \ge c \mid H_0)$



$c = Z_{\frac{\alpha}{2}}$

At the significance level α, we reject H_0 in favor of H_a if $|Z_0| \ge Z_{\frac{\alpha}{2}}$.

(For example, $\alpha = 0.05$, $|Z_0| \ge 1.96$)

Scenario 2 : Inference on one population mean μ (large sample, any population – typically not normal, σ^2 is known or unknown)

Sample is usually called ***large sample*** if $n \geq 30$

Pivotal Quantity : $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \overset{\cdot}{\sim} N(0,1)$ or $Z = \frac{\overline{X} - \mu}{s / \sqrt{n}} \overset{\cdot}{\sim} N(0,1)$

The rest is the same as Scenario 1.

Scenario 3 : Inference on one population mean μ (normal population, σ^2 is unknown)

Suppose we have a sample $X_1, X_2, \cdots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, σ^2 is unknown

First we review how to derive the pivotal quantity $T = \frac{\overline{X} - \mu}{S / \sqrt{n}} \sim t_{n-1}$

1. Point Estimator for μ : $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$

   $\overline{X}$ is ***NOT*** a pivotal quantity since σ^2 is unknown.

2. $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$

   This is also ***NOT*** a pivotal quantity since σ is unknown.

3. **Theorem.** Sample from normal population $Z \sim N(0,1)$

   $$W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Definition. $T = \frac{Z}{\sqrt{W / (n-1)}} \sim t_{n-1} \Rightarrow T = \frac{\overline{X} - \mu}{S / \sqrt{n}} \sim t_{n-1}$ (Z and W are independent.)

$\therefore T$ is a pivotal quantity for μ

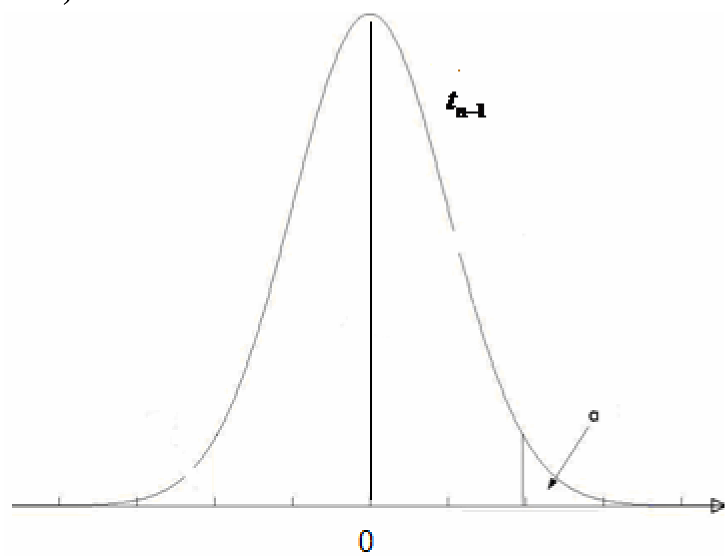
4. Now we are ready to derive the test for each pair of hypotheses

A. $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \succ \mu_0 \end{cases}$

Test Statistic : $T_0 = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \overset{H_0}{\sim} t_{n-1}$

$\alpha = P(\text{Reject } H_0 \mid H_0) = P(T_0 \geq c \mid H_0 : \mu = \mu_0)$

(α is usually 0.05.)



$c = t_{n-1,\alpha}$

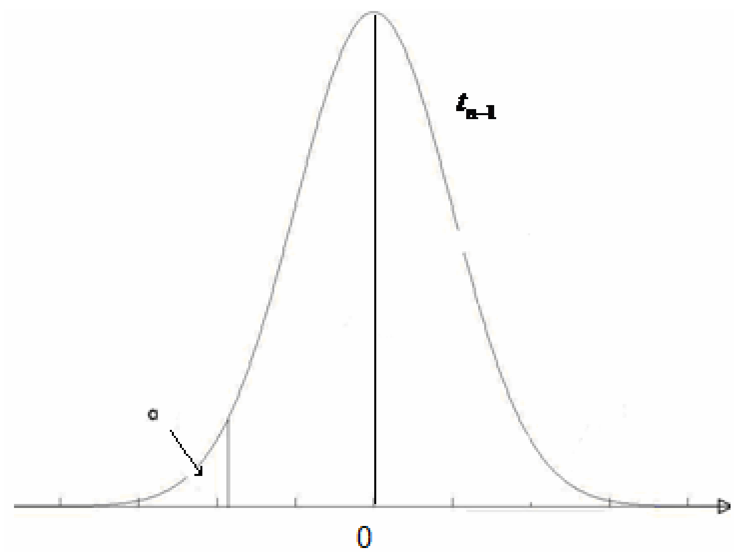
At the significance level α , we reject H_0 in favor of H_a if $T_0 \geq t_{n-1,\alpha}$

B. $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \prec \mu_0 \end{cases}$

Test Statistic : $T_0 = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \overset{H_0}{\sim} t_{n-1}$

$\alpha = P(\text{Reject } H_0 \mid H_0) = P(T_0 \leq c \mid H_0 : \mu = \mu_0)$

Reject H_0 in favor of H_a if $T_0 \leq -t_{n-1,\alpha}$



$c = -t_{n-1,\alpha}$

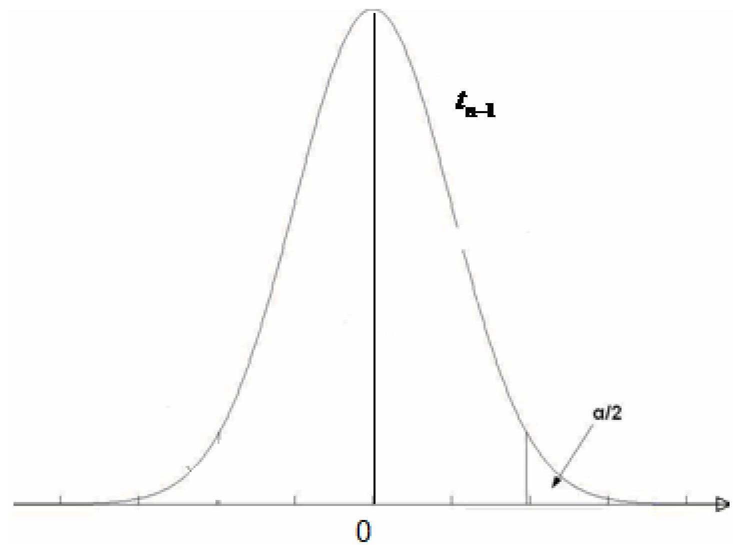
C. $\begin{cases} H_0 : \mu = \mu_0 \\ H_a : \mu \neq \mu_0 \end{cases}$

Test Statistic : $T_0 = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} \overset{H_0}{\sim} t_{n-1}$

$$\begin{aligned} \alpha &= P(\text{Reject } H_0 \mid H_0) \\ &= P(|T_0| \geq c \mid H_0) \\ &= 2 \times P(T_0 \geq c \mid H_0) \end{aligned}$$

$$\Rightarrow \frac{\alpha}{2} = P(T_0 \geq c \mid H_0)$$

At the significance level α , we reject H_0 in favor of H_a if $|T_0| \geq t_{n-1,\alpha/2}$



$c = t_{n-1,\alpha/2}$

Example. Prospective salespeople for an encyclopedia company are now offered a sales training program. Previous data indicate that the average number of sales per month for those who do not participate is 33. To determine whether the training program is effective, a random sample of 35 new employees is given the sales training and then sent.

One month later, the mean and standard deviation for the number of encyclopedia sold are 35 and 8.4, respectively. Do the data present sufficient evidence to indicate that the training program enhances sales? Use $\alpha = 0.05$

Solution.

This is Scenario 2: large sample ($n \geq 30$)

Pivotal Quantity: $Z = \dfrac{\overline{X} - \mu}{S/\sqrt{n}} \sim N(0,1)$

$$\begin{cases} H_0 : \mu = 33 \\ H_a : \mu \succ 33 \end{cases}$$

Test Statistic: $Z_0 = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} = \dfrac{35 - 33}{8.4/\sqrt{35}} = 1.41$

At $\alpha = 0.05$, we would reject H_0 in favor of H_a if $Z_0 \geq Z_{0.05} = 1.645$

$\therefore$ We do not reject H_0

If you are told the population is normal, then you ***should*** use the t-test in Scenario 3 – because the t-test is exact while the z-test is approximate in this situation. The exact test is always preferred over the approximate test.

Example. Suppose that the population is normal in the above example.

Solution.

$$\begin{cases} H_0 : \mu = 33 \\ H_a : \mu \succ 33 \end{cases}$$

Test Statistic : $T_0 = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}} = \dfrac{35 - 33}{8.4/\sqrt{35}} = 1.41$

At $\alpha = 0.05$, we would reject H_0 in favor of H_a if $T_0 \geq t_{n-1,\alpha} = t_{34,.05} = 1.69 \Leftarrow$ exact test

Because $T_0 = 1.41$ is not greater than 1.69, we cannot reject H_0

*** Please read the entire Chapter 7. ***