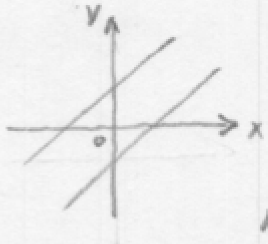
(b):



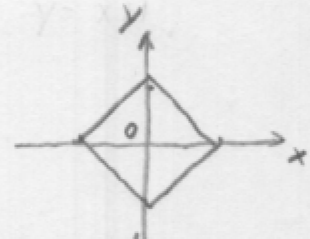
Faces: two sides.

y = x − 1

and y = x − 1

no extreme points;

(c):



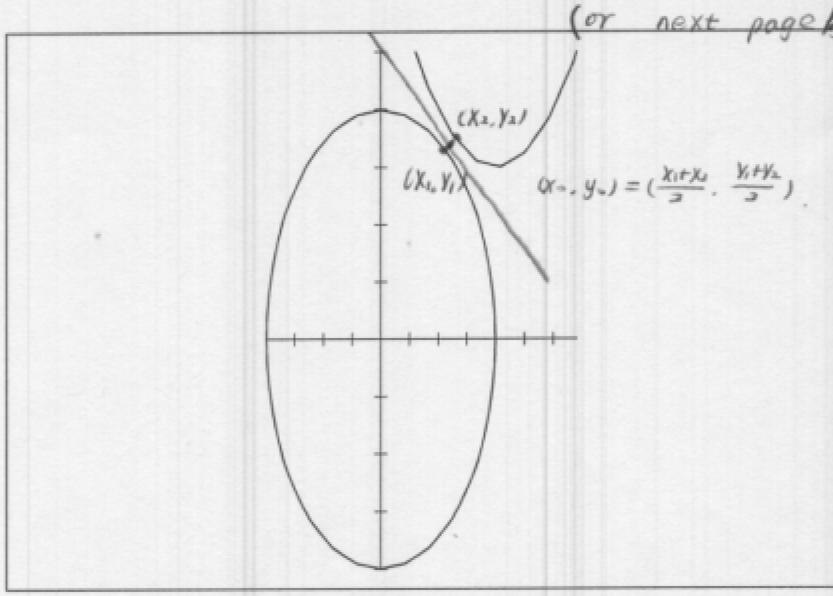
Faces: four sides.

extreme points, (0, ±1), (±1, 0).

Mathematics 120
Spring 2006

Homework Assignment No. 9

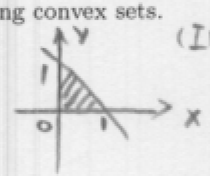
1. Find a line separating the open convex set $\left\{[x,y] : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 < 1\right\}$ from the open convex set $\left\{[x,y] : y > x^2 - 4x + 7\right\}$: Follow discussion set #4, prob. 1(b); (or next page by following the hints)



(Hint: Introduce new coordinates to "unsqueeze" the ellipse: $\xi = \frac{x}{2}$, $\eta = \frac{y}{4}$. Solve the problem in the new coordinate system. Then translate the solution back into the $x - y$ -coordinate system.)

2. Find the all faces and extreme points of the following convex sets.

(a) $\left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \Re^2 : 0 \le x, 0 \le y, x + y \le 1\right\}$ ⇒



(I) faces: three sides &

① x=0, 0≤y≤1

② y=0, 0≤x≤1

③ y=1−x, 0≤x≤1.

(II): extreme points,

①: (0,0)

②: (0,1)

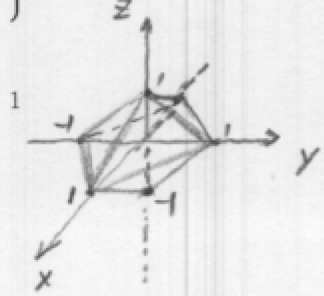
③ (1,0).

(b) $\left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \Re^2 : -1 \le x - y \le 1\right\}$

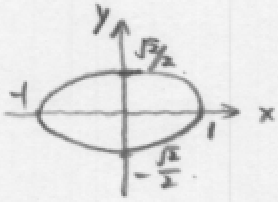
(c) $\left\{\begin{bmatrix} x \\ y \end{bmatrix} : |x| + |y| \le 1\right\}$

(d) $\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : |x| + |y| + |z| \le 1\right\}$

(d): extreme points: (±1, 0, 0),
(0, ±1, 0),
(0, 0, ±1);



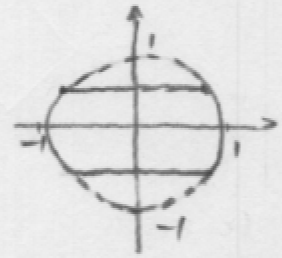
(e):



Faces: $\{x\}$. $x \in$ boundary

extereme points: $x \in$ boundary;

(f). Faces:



two sides: $y = \pm\frac{1}{2}$, $-\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2}$;

and the boundary points;

(e) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + 2y^2 \le 1 \right\}$

(f) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \text{ and } |y| \le \frac{1}{2} \right\}$

3. Let $T : \Re^n \to \Re^m$ be a linear map, and let $C \subseteq \Re^m$ be a convex set.
Prove: If $F \subseteq C$ is a face of C, then $T^{-1}(F)$ is a face of $T^{-1}(C)$.

#3: (I). Since $F \subseteq C$ is a face of C, F is convex.

T is linear, then $T^{-1}(F)$ is also convex. (for details, go to H.W. set 8, prob #5)

(II): Given any $a, b \in T^{-1}(C)$, $\lambda a + (1-\lambda) b \in T^{-1}(F)$, need to show $a, b \in T^{-1}(F)$; $(0 < \lambda < 1)$

$\Downarrow$ $T(a), T(b) \in C$. $\quad$ $\Downarrow$ $T(\lambda a + (1-\lambda) b) \in F$

$\Downarrow$

$\lambda T(a) + (1-\lambda) T(b) \in F$.

Since F is a face of C,

then $T(a), T(b) \in F$,

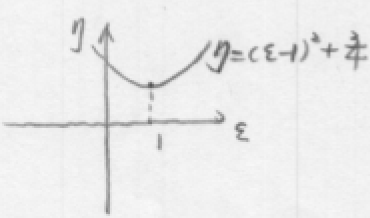
$\Rightarrow a, b \in T^{-1}(F)$;

#1: take $\begin{cases} \varepsilon = \frac{x}{2} \\ \eta = \frac{y}{4} \end{cases} \Rightarrow \begin{cases} x = 2\varepsilon \\ y = 4\eta. \end{cases}$

$(\frac{x}{2})^2 + (\frac{y}{4})^2 < 1 \Rightarrow \varepsilon^2 + \eta^2 < 1$

$y > x^2 - 4x + 7 \Rightarrow 4\eta > (2\varepsilon)^2 - 4 \cdot 2\varepsilon + 7$

$\Rightarrow \eta > (\varepsilon - 1)^2 + \frac{3}{4}$



Now consider the mini. of $d(\varepsilon, \eta) = \varepsilon^2 + \eta^2$

$= \varepsilon^2 + [(\varepsilon-1)^2 + \frac{3}{4}]^2$.

$\Rightarrow \frac{\partial d(\varepsilon, \eta)}{\partial \varepsilon} = 2\varepsilon + 2[(\varepsilon-1)^2 + \frac{3}{4}] \cdot 2(\varepsilon - 1) = 0$

$\Rightarrow 4\varepsilon^3 - 12\varepsilon^2 + 17\varepsilon - 7 = 0$;

$\Rightarrow \ldots$