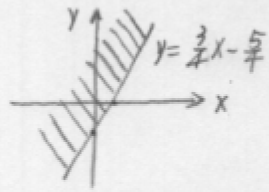
Mathematics 120
Spring 2006

Homework Assignment No. 8



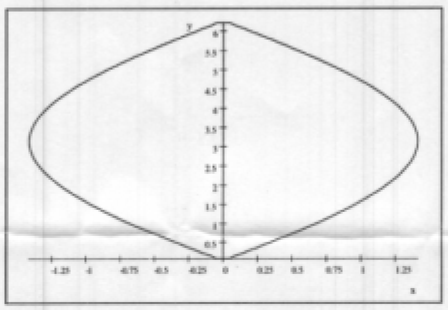
1. Plot the following half-spaces of $\Re^2$: (#a):

   (a) The half space given by $3x - 4y \leq 5$ $\Rightarrow y \geq \frac{3}{4}x - \frac{5}{4}$.

   (b) The half space given by $4x + 5y + 5 \leq 0$

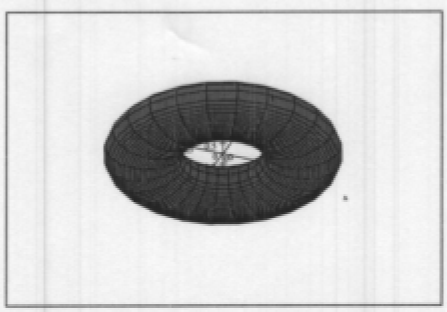
2. Which of the following sets is convex? You may assume that the graphs are correct!

   (a) The set bounded by $x^2 + \cos y = 1$:



yes.

   (b) The set bounded by $y^2 + \left(\sqrt{x^2 + z^2} - 2\right)^2 = 1$



No

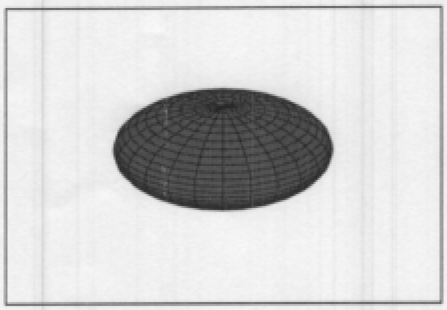
   (c) The set bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $\Rightarrow$ yes.

#5. "⇐" let $y = Ax = Tx + b_0 \Rightarrow x = T^{-1}(y - b_0)$ i.e. any element in $A^{-1}(D)$ is of the form: $T^{-1}(y-b_0)$ for some $y \in D$.

let $y_1 = Tx_1 + b_0$, $y_2 = Tx_2 + b_0 \Rightarrow x_1 = T^{-1}(y_1 - b_0)$, $x_2 = T^{-1}(y_2 - b_0)$

$y_1, y_2 \in D$ $\Rightarrow rx_1 + (1-r)x_2 = T^{-1}(ry_1 + (1-r)y_2 - b_0)$.

i.e. $ry_1 + (1-r)y_2 - b_0 = T(rx_1 + (1-r)x_2))$

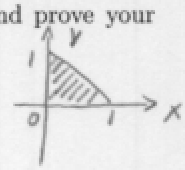


$\Rightarrow ry_1 + (1-r)y_2 = T(rx_1 + (1-r)x_2) + b_0$
$= A(rx_1 + (1-r)x_2)$.

$\Rightarrow$: $rx_1 + (1-r)x_2 \in A^{-1}(D)$

Since $ry_1 + (1-r)y_2 \in D$.
D is convex, $y_1, y_2 \in D$;

3. Which of the following sets is convex? Sketch the sets and prove your answers!



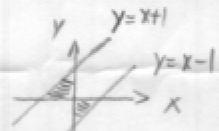
(a) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \Re^2 : 0 \le x, 0 \le y, x + y \le 1 \right\}$; Yes

(b) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \Re^2 : x = 0 \text{ or } y = 0 \right\}$; No

(c) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \Re^2 : 1 \le x^2 + y^2 \le 4 \right\}$; No.

(d) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \Re^2 : -1 \le x - y \le 1 \right\}$ Yes

(e) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \Re^2 : xy \le 0 \text{ and } -1 \le x - y \le 1 \right\}$



NOT convex,

4. Find the convex hulls of the following subsets of $\Re^n$. Sketch!

intersection of all convex subsets of A.

or: the smallest convex set containing A.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ "see Prop. 2 on p52"

(b) $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\}$

(d) $\{\vec{x} \in \Re^2 : x_1^2 + x_2^2 = 1\}$ $\Rightarrow$: $\{x \in R^2 : x_1^2 + x_2^2 \le 1\}$

(e) $\{\vec{x} \in \Re^2 : x_1^2 + x_2^2 \ge 1\}$ $\Rightarrow$: R^2

(f) $\{\vec{x} \in \Re^2 : |x_1| + |x_2| = 1\}$ $\Rightarrow$: $\{x \in R^2 : |x_1| + |x_2| \le 1\}$

5. An affine map from $\Re^n \to \Re^m$ is a map of the form

$$A\vec{x} = T\vec{x} + \vec{b}_0$$

where T is linear and $\vec{b}_0 \in \Re^m$ is a fixed vector. Show: If $C \subseteq \Re^n$ is a convex set, then $A(C)$ is a convex set. Similarly, if $D \subseteq \Re^m$ is a convex set, then $A^{-1}(D)$ is a convex set.

#5:

"⇒" For any two elements in $A(C)$, they are of the form Ax_1, Ax_2 for some $x_1, x_2 \in C$, need to show $rAx_1 + (1-r)Ax_2 \in A(C)$. Since $Ax_1 = Tx_1 + \vec{b}_0$, $Ax_2 = Tx_2 + \vec{b}_0$, $rA(x_1) + (1-r)Ax_2 = T(rx_1 + (1-r)x_2) + \vec{b}_0 \in A(C)$ since C is convex, $rx_1 + (1-r)x_2 \in C$ for $\forall x_1, x_2 \in C$, $0 < r < 1$;