Mathematics 120
Spring 2006

Homework Assignment No. 10

Try to understand Dissussion set 5 prob. #2, If you can do that, then you would be able to do these.

Too many here;

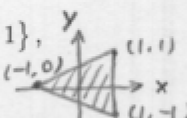
1. Which of the following functions is convex on $\Re \times \Re$?

   check Hessian matrix H(f) is positive semi-definite or not;

   (a) $x^2 + y^4$,
   (b) $x^3 + y^2$,
   (c) $x - y$
   (d) $ax + by$, where a and b are constants,
   (e) $-\ln\left(1 + x^2 + y^2\right)$.

2. For each of the convex functions from the previous problem, find the minimum and the maximum of $f(x, y)$ on the following convex sets:

   (a) $C = \left\{(x, y) : x^2 + y^2 \leq 1\right\}$,
   (b) $C = [-1, 1] \times [-1, 1]$,
   (c) $C = \left\{(x, y) \in \Re^2 : -1 \leq x + y \leq 1 \text{ and } -1 \leq x - y \leq 1\right\}$,
   (d) C is the convex hull of $\{(-1, 0), (1, 1), (1, -1)\}$. =



3. Let $C \subseteq \Re^n$ be a convex set, and let $f, g : C \to \Re$ be two convex functions. For each of the following statements, either provide a proof of the statement or construct a counter example:

   $(f+g)(\lambda x + (1-\lambda)y)$
   $= f(\lambda x + (1-\lambda)y) + g(\lambda x + (1-\lambda)y)$
   $\leq \lambda f(x) + (1-\lambda) f(y) + \lambda g(x) + (1-\lambda) g(y)$
   $= \lambda (f+g)(x) + (1-\lambda)(f+g)(y)$;

   ✓ (a) The function $f + g$ is again a convex functions. ✓

   $r \geq 0$ ⇐ ✗ (b) If $r \in \Re$ is a real number, then the function rf is convex.

   ✗ (c) The function $f \cdot g$ is convex. $f = -1$, $g = x^2$

   For the following statements, assume that $C = [a, b]$ is an interval in $\Re$.

   ✗ (d) If $f(x) > 0$ for all $x \in [a, b]$, then the function h defined by $h(x) = \frac{1}{f(x)}$ is convex. $f(x) = |x| + 1$. $x \in [-1, 1]$

   ✗ (e) If $f(x) < 0$ for all $x \in [a, b]$, then the function h defined by $h(x) = \frac{1}{f(x)}$ is convex. $f(x) = x^2 - 2$. $x \in [-1, 1]$

4. Given are:

   (a) A convex set $C \subseteq \Re^n$.
   (b) A convex function $f : C \to \Re$
   (c) An interval $[a, b] \subseteq \Re$ so that $f(C) \subseteq [a, b]$

4.#(d): $g\circ f(\lambda x+(1-\lambda)y) = g[f(\lambda x+(1-\lambda)y)] \le \lambda(g\circ f)(x) + (1-\lambda)(g\circ f)(y)$

want $g\circ f(\lambda x+(1-\lambda)y) \le \lambda(g\circ f)(x) + (1-\lambda)(g\circ f)(y)$

If g is nondecreasing, since $f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y)$

then $g[f(\lambda x+(1-\lambda)y)] \le g[\lambda f(x)+(1-\lambda)f(y)]$

$\le \lambda(g\circ f)(x)+(1-\lambda)(g\circ f)(y)$

by convexity of g.

NOT always!

△ (d) A convex function $g : [a,b] \to \Re$.
Is the composition $g \circ f$ again convex? What happens if $g(x) = e^x$? ← yes
Find a necessary condition under which you can be sure that $g \circ f$ is convex. ⟹ g is nondecreasing.

5. Let $f(x,y,z) = x^2+y^2+z^2+xy+xz+yz-8x+9y+10z$ and let $h(x,y,z) = \exp(x^2+y^2+z^2+xy+xz+yz-3x-5z)$.

(a) Show that the function $f(x,y,z)$ is convex. $H(f)$ positive semi-definite;

(b) Prove that $h(x,y,z) = \exp(x^2+y^2+z^2+xy+xz+yz-3x-5z)$ is convex (use 4d)

(c) Find the absolute minimum and the absolute maximum of $h(x,y,z)$ on the set $\{[x,y,z] : |x|+|y|+|z| \le 10\}$

5 (#b): let $g(x) = x^2+y^2+z^2+xy+xz+yz-3x-5z)$

Show $h(x,y,z) = e^{g(x)}$ is convex by showing $g(x)$ is convex (Hessian is positive semi-definite)

then use #4(d)

(#c): $Dh = (e^{g(x)}(2x+y+z-3),\ e^{g(x)}(2y+x+z),\ e^{g(x)}(x+y+2z-5)) = \vec{0}$

$$\Rightarrow \begin{cases} 2x+y+z=3 & ① \\ x+2y+z=0 & ② \\ x+y+2z=5 & ③ \end{cases}$$

$+$

$4x+4y+4z=8$

$\Rightarrow x+y+z=2$. ④

①−④ ⟹ $x_0=1$

②−④ ⟹ $y_0=-2$

③−④ ⟹ $z_0=3$

and $|x_0|+|y_0|+|z_0| = 6 < 10$. so Global min. $= h(1,-2,3)$ $= \cdots$ (by yourself).

For the absolute maximum, just compare the values at those 6 extreme points: $(\pm10,0,0)$, $(0,\pm10,0)$, $(0,0,\pm10)$ to get the maximum of $h(x)$, so to get maximum of $e^{g(x)}$;