Normal Mixed Models

Mixed models with normally distributed errors. Previously, we defined several mixed models using notation to suit each situation. Now we will start using a more general notation using matrix notation.

Model Definition

All types of mixed models can be defined using general matrix notation.

The fixed effects model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_p x_{ip} + e_i$$

var(e_i)=σ^2

If we have *n* observations, we would write the model as follows:

$$y_1 = \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + ... + \beta_p x_{1p} + e_1$$
$$y_2 = \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} + ... + \beta_p x_{2p} + e_2$$
$$y_3 = \beta_0 + \beta_1 x_{31} + \beta_2 x_{32} + ... + \beta_p x_{3p} + e_3$$
$$y_4 = \beta_0 + \beta_1 x_{41} + \beta_2 x_{42} + ... + \beta_p x_{4p} + e_4$$
.
.
$$y_n = \beta_0 + \beta_1 x_{n1} + \beta_2 x_{n2} + ... + \beta_p x_{np} + e_n$$

$$\text{var}(e_1) = \sigma^2$$
$$\text{var}(e_2) = \sigma^2$$
$$\text{var}(e_3) = \sigma^2$$
$$\text{var}(e_4) = \sigma^2$$
.
.
.
$$\text{var}(e_n) = \sigma^2$$

$$X = \begin{bmatrix} 1 & x_{11} & x_{12} & . & x_{1p} \\ 1 & x_{21} & x_{22} & . & x_{2p} \\ 1 & x_{31} & x_{32} & . & x_{3p} \\ 1 & x_{41} & x_{42} & . & x_{4p} \\ . & . & . & . & . \\ 1 & x_{n1} & x_{n2} & . & x_{np} \end{bmatrix} \quad \text{(columns: } \beta_0 \; \beta_1 \; \beta_2 \; . \; \beta_p \text{)}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ . \\ y_n \end{bmatrix} \qquad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ . \\ e_n \end{bmatrix}$$

$$V = \mathrm{var}(y) = \begin{bmatrix} \sigma^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2 \end{bmatrix} = \sigma^2 \mathrm{I}$$

Expressed in matrix notation

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$$

$$\mathbf{V} = \mathbf{var(y)} = \boldsymbol{\sigma}^2\mathbf{I}$$

The parameters in **β** can incompass several variables, both covariate and factor effects. The factor effects can be either qualitative or categorical, where one or more observations belong to one of several classes. For a covariate, only one parameter is needed, while one parameter is needed for each value of the fixed effect.

We will go through an example using this notation using data from **SAS for Linear Models (2002)**

There are 5 treatments. The response varable is final weight. An initial pre-treatment weight was also recorded. The animals were confined to two pastures.

Treatment	Pasture	Initial Weight	Final Weight
1	A	27.2	32.6
2	A	28.6	33.8
2	A	26.8	31.7
3	A	22.4	29.1
3	A	23.2	28.9
4	A	30.3	36.4
5	A	20.4	24.6
1	B	32.0	36.6
1	B	26.8	31.0
2	B	26.5	30.7
3	B	28.6	35.2
4	B	29.3	35.0
4	B	21.8	27.0
5	B	19.6	23.4
5	B	18.1	21.8

y=(32.6, 33.8,31.7, 29.1,28.9, 36.4,24.6,36.6,31.0,30.7,35.2,35.0,27.0,23.4,21.8)'.

If we want to fit an ANCOVA model (ignoring pastures) with treatment and pre-treatment initial weight, the **X** or design matrix would be:

$$X = \begin{array}{c} \begin{matrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \end{matrix} \\ \begin{bmatrix} 1 & 27.2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 28.6 & 0 & 1 & 0 & 0 & 0 \\ 1 & 26.8 & 0 & 1 & 0 & 0 & 0 \\ 1 & 22.4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 23.2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 30.3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 20.4 & 0 & 0 & 0 & 0 & 1 \\ 1 & 32.0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 26.8 & 1 & 0 & 0 & 0 & 0 \\ 1 & 26.5 & 0 & 1 & 0 & 0 & 0 \\ 1 & 28.6 & 0 & 0 & 1 & 0 & 0 \\ 1 & 29.3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 21.8 & 0 & 0 & 0 & 1 & 0 \\ 1 & 19.6 & 0 & 0 & 0 & 0 & 1 \\ 1 & 18.1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that the design matrix is *overparameterized*. There are linear dependencies among the columns. The sum of the last 5 columns equals the first column.

The mixed model

The mixed model has both fixed and random effects.

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} +$$
$$\mu_1 z_{i1} + \mu_2 z_{i2} + \ldots + \mu_q z_{iq} + e_i$$

where we have p fixed effect parameters and q random effect parameters.

This can be expressed in matrix notation.

$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\mu} + \mathbf{e}$

y, X, β, and **e** are the same as for the fixed effect model.

$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_q)'$ = random effects/coefficient parameters

Z is the design matrix relating **y** to **μ**. It is specified the same way as **X**, except that there is no intercept term (β_0).

For our example, if we treat pasture as a random effect,

$$\begin{matrix} & \mu_A & \mu_B \end{matrix}$$
$$Z = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

If we include both a pasture effect and a pasture*treatment effect as random, the **Z** matrix would have 12 columns, two for the pasture parameters and 10 for the pasture*treatment parameters.

$$
\begin{array}{c}
\\
Z=
\end{array}
\begin{array}{cccccccccccc}
\mu_A & \mu_B & \mu_{A1} & \mu_{A2} & \mu_{A3} & \mu_{A4} & \mu_{A5} & \mu_{B1} & \mu_{B2} & \mu_{B3} & \mu_{B4} & \mu_{B5} \\
\end{array}
$$

$$
Z=\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Note that this matrix is overparameterized. The sum of columns 3-7 (or all of the pasture A*treatment) is column 1 (Pasture A) and the sum of columns 8-12 (or all of the pasture B*treatment) is column 2 (Pasture B).

Covariance Matrix V

For the fixed effect model, the observations were uncorrelated and had equal variance (σ^2), so $V=\sigma^2 I$, where I was an nxn identity matrix.

Include random effects in the model results in correlated observations.

The covariance of **y** is designated as var(**y**)=**V** where

$\mathbf{V} = \text{var}(\mathbf{X\beta}+\mathbf{Z\mu}+\mathbf{e})$

We assume that the random effects and the residuals are uncorrelated.

$\mathbf{V} = \text{var}(\mathbf{X\beta})+\text{var}(\mathbf{Z\mu})+\text{var}(\mathbf{e})$

$\boldsymbol{\beta}$ are the fixed effects so the variance of $\mathbf{X\beta}$ is 0.

We'll define the var(**e**)=**R**=σ^2I.
Because **Z** is a matrix of constants, we can pull it out.

$\mathbf{V} = \mathbf{Z}\text{var}(\boldsymbol{\mu})\mathbf{Z}'+\mathbf{R}$

We'll define var($\boldsymbol{\mu}$)=**G.**

V=ZGZ'+R

Structures for the **G** and **R** matrices are dependent on the model
- -random effects models
- -random coefficients models
- -covariance pattern models

Covariance structure for random effects models

The G matrix

G is a q x q matrix where q is the number of random effect parameters. G is always diagonal in a random effects model if the random effects are assumed uncorrelated. In our example, if just the pasture effect was fitted as random

$$G = \begin{bmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_p^2 \end{bmatrix}$$

where σ_p^2 is the pasture variance component.

If both pasture and pasture*treatment effects are fitted as random

$$G = \begin{bmatrix}
\sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma_{pt}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma_{pt}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma_{pt}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma_{pt}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_{pt}^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{pt}^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{pt}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{pt}^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{pt}^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{pt}^2
\end{bmatrix}$$

where σ_{pt} is the pasture*treatment variance component.

The R matrix

Residuals are uncorrelated.

$R = \sigma^2 I$

$$R = \begin{bmatrix}
\sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2
\end{bmatrix}$$

The V matrix

V=ZGZ’+R

In our example, if just pasture effects are fitted as random we have the following:

$$Z = \begin{array}{c} \begin{matrix} \mu_A & \mu_B \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \quad G = \begin{bmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_p^2 \end{bmatrix}$$

which gives us the following

$$ZGZ' = \begin{bmatrix}
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2
\end{bmatrix}$$

which is a block diagonal matrix where each block is the dimension of each block corresponds to the number of observations in each pasture category.

So summing **ZGZ'+R** we get the following total variance matrix

$$V = \begin{bmatrix}
\sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma^2
\end{bmatrix}$$

Note that this also has a block diagonal form, with the covariances among observations from the same pasture equal to σ_p^2 and the variance for each observation equal to $\sigma_p^2+\sigma^2$.

If both pasture and pasture*treatment are fitted as random effects

$$ZGZ'=\begin{bmatrix}
\sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2+\sigma_{pt}^2
\end{bmatrix}$$

$$V=\begin{bmatrix}
\theta & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \theta & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \theta & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \theta & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \theta & \sigma_p^2 & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \theta & \sigma_p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2+\sigma_{pt}^2 & \theta & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \theta & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \theta & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \theta & \sigma_p^2+\sigma_{pt}^2 & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \theta & \sigma_p^2 & \sigma_p^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \theta & \sigma_p^2+\sigma_{pt}^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2 & \sigma_p^2+\sigma_{pt}^2 & \theta
\end{bmatrix}$$

where $\theta=\sigma_p^2+\sigma_{pt}^2+\sigma^2$. So **V** is still block diagonal, but with a more complicated structure.

Random coefficients model covariance structure

For this model, we are going to go back to the mult-center study from the textbook and use the data from the first three patients, whose data are presented below.

Patient	Treatment	Time(days)
1	A	t_{11}
1	A	t_{12}
1	A	t_{13}
1	A	t_{14}
2	B	t_{21}
2	B	t_{22}
3	A	t_{31}
3	A	t_{32}
3	A	t_{33}

If both patient and patient*time were fitted as random coeffients, we would need 6 random coefficients. Each patient would need both an intercept and slope ($\mu_{p,1}$, $\mu_{pt,1}$, $\mu_{p,2}$, $\mu_{pt,2}$, $\mu_{p,3}$, $\mu_{pt,3}$).

$$\begin{matrix} \mu_{p,1} & \mu_{pt,1} & \mu_{p,2} & \mu_{pt,2} & \mu_{p,3} & \mu_{pt,3} \end{matrix}$$

$$Z = \begin{bmatrix} 1 & t_{11} & 0 & 0 & 0 & 0 \\ 1 & t_{12} & 0 & 0 & 0 & 0 \\ 1 & t_{13} & 0 & 0 & 0 & 0 \\ 1 & t_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t_{21} & 0 & 0 \\ 0 & 0 & 1 & t_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_{31} \\ 0 & 0 & 0 & 0 & 1 & t_{32} \\ 0 & 0 & 0 & 0 & 1 & t_{33} \end{bmatrix}$$

The R matrix

The residuals are uncorrelated.

$$R = \begin{bmatrix} \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 \end{bmatrix}$$

The G matrix

The patient effects (intercepts) are correlated with the random patient*time effects (slopes). Again, correlations are only going to occur within patients. The dimensions of **G** are the number of parameters (6).

$$G=\begin{bmatrix} \sigma_p^2 & \sigma_{p,pt} & 0 & 0 & 0 & 0 \\ \sigma_{p,pt} & \sigma_{pt}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_p^2 & \sigma_{p,pt} & 0 & 0 \\ 0 & 0 & \sigma_{p,pt} & \sigma_{pt}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_{p,pt} \\ 0 & 0 & 0 & 0 & \sigma_{p,pt} & \sigma_{pt}^2 \end{bmatrix}$$

where

σ_p^2 =patient variance component

σ_{pt}^2 = patient*time variance component

$\sigma_{p,pt}$ = covariance between the random coefficients

The V matrix

V=ZGZ'+R

$$\begin{matrix} \mu_{p,1} & \mu_{pt,1} & \mu_{p,2} & \mu_{pt,2} & \mu_{p,3} & \mu_{pt,3} \end{matrix}$$

$$Z=\begin{bmatrix} 1 & t_{11} & 0 & 0 & 0 & 0 \\ 1 & t_{12} & 0 & 0 & 0 & 0 \\ 1 & t_{13} & 0 & 0 & 0 & 0 \\ 1 & t_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t_{21} & 0 & 0 \\ 0 & 0 & 1 & t_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_{31} \\ 0 & 0 & 0 & 0 & 1 & t_{32} \\ 0 & 0 & 0 & 0 & 1 & t_{33} \end{bmatrix} G=\begin{bmatrix} \sigma_p^2 & \sigma_{p,pt} & 0 & 0 & 0 & 0 \\ \sigma_{p,pt} & \sigma_{pt}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_p^2 & \sigma_{p,pt} & 0 & 0 \\ 0 & 0 & \sigma_{p,pt} & \sigma_{pt}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_p^2 & \sigma_{p,pt} \\ 0 & 0 & 0 & 0 & \sigma_{p,pt} & \sigma_{pt}^2 \end{bmatrix}$$

$$ZGZ'=\begin{bmatrix} v_{1,11} & v_{1,12} & v_{1,13} & v_{1,14} & 0 & 0 & 0 & 0 & 0 \\ v_{1,12} & v_{1,22} & v_{1,23} & v_{1,24} & 0 & 0 & 0 & 0 & 0 \\ v_{1,13} & v_{1,23} & v_{1,33} & v_{1,34} & 0 & 0 & 0 & 0 & 0 \\ v_{1,14} & v_{1,24} & v_{1,34} & v_{1,44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{2,11} & v_{2,12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{2,12} & v_{2,22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{3,11} & v_{3,12} & v_{3,13} \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{3,12} & v_{3,22} & v_{3,23} \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{3,13} & v_{3,23} & v_{3,33} \end{bmatrix}$$

where $v_{i,jk}=\sigma_p^2+(t_{ij}+t_{kj})\sigma_{p,pt}+t_{ij}t_{ik}\sigma_{pt}^2$.

$$V = \begin{bmatrix} v_{1,11}+\sigma^2 & v_{1,12} & v_{1,13} & v_{1,14} & 0 & 0 & 0 & 0 & 0 \\ v_{1,12} & v_{1,22}+\sigma^2 & v_{1,23} & v_{1,24} & 0 & 0 & 0 & 0 & 0 \\ v_{1,13} & v_{1,23} & v_{1,33}+\sigma^2 & v_{1,34} & 0 & 0 & 0 & 0 & 0 \\ v_{1,14} & v_{1,24} & v_{1,34} & v_{1,44}+\sigma^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{2,11}+\sigma^2 & v_{2,12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{2,12} & v_{2,22}+\sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{3,11}+\sigma^2 & v_{3,12} & v_{3,13} \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{3,12} & v_{3,22}+\sigma^2 & v_{3,23} \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{3,13} & v_{3,23} & v_{3,33}+\sigma^2 \end{bmatrix}$$

The covariance structure in a random coefficient model is determined by the random coefficients.

The covariance pattern model covariance structure

In covariance pattern models the covariance structure has a specific pattern directly in the **R** (or occasionly **G**) matrix. The pattern is usually dependent on a variable such as time or visit.

$$R = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & R_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & R_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & R_4 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & R_5 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & R_6 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & R_7 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Each submatrices are covariance blocks corresponding to the ith blocking effect (i.e. animal)a dimension equal to the number of repeated records for that animal. Following is a structure for animals with repeated records in a feeding trial.

Animal	**Week**
1	1
1	2
1	3
2	1
2	2
2	3
2	4
3	1
3	2

Using animal as the blocking effect

$$R = \begin{bmatrix} \sigma_1^2 & \theta_{12} & \theta_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_{12} & \sigma_2^2 & \theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_{13} & \theta_{23} & \sigma_3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1^2 & \theta_{12} & \theta_{13} & \theta_{14} & 0 & 0 \\ 0 & 0 & 0 & \theta_{12} & \sigma_2^2 & \theta_{23} & \theta_{24} & 0 & 0 \\ 0 & 0 & 0 & \theta_{13} & \theta_{23} & \sigma_3^2 & \theta_{34} & 0 & 0 \\ 0 & 0 & 0 & \theta_{14} & \theta_{24} & \theta_{34} & \sigma_4^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_1^2 & \theta_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{12} & \sigma_2^2 \end{bmatrix}$$

where there is a separate correlation for each pair of visits.

An alternative pattern known as compound symmetry is

$$R = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \rho\sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho\sigma^2 & \rho\sigma^2 & \sigma^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \rho\sigma^2 & \rho\sigma^2 & \rho\sigma^2 & 0 & 0 \\ 0 & 0 & 0 & \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \rho\sigma^2 & 0 & 0 \\ 0 & 0 & 0 & \rho\sigma^2 & \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & 0 & 0 \\ 0 & 0 & 0 & \rho\sigma^2 & \rho\sigma^2 & \rho\sigma^2 & \sigma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma^2 & \rho\sigma^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

where ρ is the correlation between observations on the same patient.