**Incomplete Block Designs**

With complete block designs, all treatments are represented in each block of the experiment. Sometimes you may have a large number of treatments that you want to compare and the blocks that you have available can only handle a limited number of these treatments. Incomplete block designs are used in these situations. These designs are partially confounded designs that allow some information an all contrasts, as long as there is adequate replication. In general, each class may be estimated with a different amount of information, although there are designs that have some type of balance. One type of incomplete block design that has some balance is the balanced incomplete block design (BIB). In general, an incomplete block design can be thought of as a two-factor experiment (block and treatment) with unequal numbers and no interaction between the block and treatment factor, leading to the following linear model:

$$y_{ijg} = \mu + \tau_i + \beta_j + e_{ijg}$$

$\tau_i$ = ith treatment effect where i=1,t
$\beta_j$ = jth block effect where j=1,b, where blocks can be either fixed or random
g = $n_{ij}$ where $n_{ij}$ =0 if the ith treatment doesn't appear in the jth block
$n_{ij}$ =1 if the ith treatment does appear in the jth block
$e_{ijg} \sim (0,\sigma^2)$

**Balanced incomplete block design (BIB)**

Balanced incomplete block (BIB) design is the first design we'll consider. With incomplete block designs, in general, each treatment difference may be estimated with a different amount of information. With a BIB design every treatment difference is estimated with the same amount of information (estimated with the same variance).

A BIB meets the following conditions when there are *t* treatments with *b* blocks:

1. Each block contains k<t treatments
2. Each treatment appears in exactly *r* blocks
3. Every pair of treatments occurs together in *l* blocks.

These conditions are similar to those we saw with the variance balanced cross-over designs. Also, from the design of the study $l<r<b$.

For example, if we have 4 treatments (t) and blocks of size 3 (k), means that the BIB that we can construct would have *r*=2 and *l*=1, leading to the following design:

| **Block** | **Treatments** | | |
|---|---|---|---|
| 1 | A | B | C |
| 2 | A | B | D |
| 3 | A | C | D |
| 4 | B | C | D |

If the blocks available were of size=2, then we could construct the following BIB design:

| Block | Treatments | |
|---|---|---|
| 1 | A | B |
| 2 | C | D |
| 3 | A | C |
| 4 | B | D |
| 5 | A | D |
| 6 | B | C |

Note for both of these designs, all treatments appear an equal number of times and appear in the same number of blocks and all pairs of treatments appear the same number of times.

We'll now go through the example of a BIB design presented in "SAS for Linear Models". Following are the data using the BIB design presented in the second table:

| Obs | trt | blk | y |
|---|---|---|---|
| 1 | 1 | 1 | 1.2 |
| 2 | 2 | 1 | 2.7 |
| 3 | 3 | 2 | 7.1 |
| 4 | 4 | 2 | 8.6 |
| 5 | 1 | 3 | 7.1 |
| 6 | 3 | 3 | 9.7 |
| 7 | 2 | 4 | 8.8 |
| 8 | 4 | 4 | 15.1 |
| 9 | 1 | 5 | 9.7 |
| 10 | 4 | 5 | 17.4 |
| 11 | 2 | 6 | 13.0 |
| 12 | 3 | 6 | 16.6 |

Note that Treatment 1 occurs only in blocks 1, 3, and 5. We'll first do an analysis treating the blocks as a fixed effect using the following program:

```
proc glm;
class blk trt;
model y=trt blk / e1 ss3;
means trt blk;
lsmeans trt /stderr pdiff cl;
run;
```

Note that with the e1 option, we are requesting the Type I estimable functions.

We get the following output

Dependent Variable: y

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 8 | 281.1275000 | 35.1409375 | 40.82 | 0.0056 |
| Error | 3 | 2.5825000 | 0.8608333 | | |
| Corrected Total | 11 | 283.7100000 | | | |

| R-Square | Coeff Var | Root MSE | y Mean |
|---|---|---|---|
| 0.990897 | 9.516011 | 0.927811 | 9.750000 |

| Source | DF | Type I SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| trt | 3 | 102.2566667 | 34.0855556 | 39.60 | 0.0065 |
| blk | 5 | 178.8708333 | 35.7741667 | 41.56 | 0.0057 |

| Source | DF | Type III SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| trt | 3 | 59.0175000 | 19.6725000 | 22.85 | 0.0144 |
| blk | 5 | 178.8708333 | 35.7741667 | 41.56 | 0.0057 |

The type I sums of squares (SS) for treatment are unadjusted and based on the ordinary treatment means. So this SS contains both the treatment and block differences. The type III SS for treatment is adjusted for block, so the mean square (MS) for treatment measures the difference between treatment means and random error.

Following are the estimable functions for both Type I and Type III which further illustrate differences between the Type I and Type III. First, we will look at the Type I estimable function looking at the effect of treatment 1 or the difference between treatment 1 mean and the mean from the mean of all treatments or

TRT1-1/4(TRT1+TRT2+TRT3+TRT4)=3/4TRT1-1/4TRT2-1/4TRT3-1/4TRT4

We can use the Type I estimable functions to get the above by defining the following:

L2=3/4
L3=-1/4
L4=-1/4

-L2-L3-L4=-3/4+1/4+1/4=-1/4

Note that the Trt1 effect also is a contrast between blocks 1, 3, and 5 which contain Treatment 1 and blocks 2, 4, and 6 which do not. Note that the Type III estimable functions do not include block effects.

Type I Estimable Functions

| | --------------Coefficients-------------- | |
|---|---|---|
| Effect | trt | Trt1 effect |

```
Intercept       0                       0
trt       1     L2                      +.75
trt       2     L3                      -.25
trt       3     L4                      -.25
trt       4     -L2-L3-L4               -.25
blk       1     0.3333*L2+0.3333*L3     .167
blk       2     -0.3333*L2-0.3333*L3    -.167
blk       3     0.3333*L2+0.3333*L4     .167
blk       4     -0.3333*L2-0.3333*L4    -.167
blk       5     -0.3333*L3-0.3333*L4    .167
blk       6     0.3333*L3+0.3333*L4     -.167

             Type III Estimable Functions

                     --------Coefficients---------
     Effect          trt          blk

     Intercept       0            0
     trt       1     L2           0
     trt       2     L3           0
     trt       3     L4           0
     trt       4     -L2-L3-L4    0
     blk       1     0            L6
     blk       2     0            L7
     blk       3     0            L8
     blk       4     0            L9
     blk       5     0            L10
     blk       6     0            -L6-L7-L8-L9-L10
```

We can see the presence of the block effect further when we compare the expected mean squares from Type I and Type III, where the Type I trt EMS includes the block variation, while the Type III does not.

```
Source                Type I Expected Mean Square

trt                   Var(Error) + 0.6667 Var(blk) + Q(trt)
blk                   Var(Error) + 1.6 Var(blk)

Source                Type III Expected Mean Square

trt                   Var(Error) + Q(trt)
blk                   Var(Error) + 1.6 Var(blk)
```

Finally, we will look at the least squares means. These least squares means are adjusted for block effects and correspond to the Type III estimable functions.

```
                              The GLM Procedure
                              Least Squares Means
```

| trt | y LSMEAN | Standard Error | Pr > \|t\| | LSMEAN Number |
|---|---|---|---|---|
| 1 | 6.8000000 | 0.6281310 | 0.0017 | 1 |
| 2 | 7.6500000 | 0.6281310 | 0.0012 | 2 |
| 3 | 10.9250000 | 0.6281310 | 0.0004 | 3 |
| 4 | 13.6250000 | 0.6281310 | 0.0002 | 4 |

Least Squares Means for effect trt
Pr > |t| for H0: LSMean(i)=LSMean(j)

Dependent Variable: y

| i/j | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | 0.4272 | 0.0212 | 0.0052 |
| 2 | 0.4272 | | 0.0386 | 0.0076 |
| 3 | 0.0212 | 0.0386 | | 0.0620 |
| 4 | 0.0052 | 0.0076 | 0.0620 | |

| trt | y LSMEAN | 95% Confidence Limits | |
|---|---|---|---|
| 1 | 6.800000 | 4.801007 | 8.798993 |
| 2 | 7.650000 | 5.651007 | 9.648993 |
| 3 | 10.925000 | 8.926007 | 12.923993 |
| 4 | 13.625000 | 11.626007 | 15.623993 |

Least Squares Means for Effect trt

| i | j | Difference Between Means | 95% Confidence Limits for LSMean(i)-LSMean(j) | |
|---|---|---|---|---|
| 1 | 2 | -0.850000 | -3.802709 | 2.102709 |
| 1 | 3 | -4.125000 | -7.077709 | -1.172291 |
| 1 | 4 | -6.825000 | -9.777709 | -3.872291 |
| 2 | 3 | -3.275000 | -6.227709 | -0.322291 |
| 2 | 4 | -5.975000 | -8.927709 | -3.022291 |
| 3 | 4 | -2.700000 | -5.652709 | 0.252709 |

NOTE: To ensure overall protection level, only probabilities associated with
pre-planned comparisons should be used.

The differences between the least squares means give us the "intra-block" comparisons of treatments. However, there is also inter-block information about differences between the treatment means contained in the means that is not used in the intra-block comparison, that is called inter-block information. Treating block as a random effect allows us to get the combined inter- and intra-block information about differences between treatment means. The following SAS program performs this analysis:

```
proc mixed data=bib;
class blk trt;
model y=trt/ddfm=kr;
random blk;
lsmeans trt/ pdiff cl;
run;
```

With the following selected results:

Covariance Parameter Estimates

| Cov Parm | Estimate |
|---|---|
| blk | 17.8543 |
| Residual | 0.8518 |

The Mixed Procedure

Type 3 Tests of Fixed Effects

| Effect | Num DF | Den DF | F Value | Pr > F |
|---|---|---|---|---|
| trt | 3 | 3.13 | 22.92 | 0.0125 |

Least Squares Means

| Effect | trt | Estimate | Standard Error | DF | t Value | Pr > \|t\| | Alpha |
|---|---|---|---|---|---|---|---|
| trt | 1 | 6.7724 | 1.8357 | 5.96 | 3.69 | 0.0104 | 0.05 |
| trt | 2 | 7.6678 | 1.8357 | 5.96 | 4.18 | 0.0059 | 0.05 |
| trt | 3 | 10.9322 | 1.8357 | 5.96 | 5.96 | 0.0010 | 0.05 |
| trt | 4 | 13.6276 | 1.8357 | 5.96 | 7.42 | 0.0003 | 0.05 |

Least Squares Means

| Effect | trt | Lower | Upper |
|---|---|---|---|
| trt | 1 | 2.2723 | 11.2725 |
| trt | 2 | 3.1678 | 12.1679 |
| trt | 3 | 6.4321 | 15.4323 |
| trt | 4 | 9.1275 | 18.1277 |

Differences of Least Squares Means

| Effect | trt | _trt | Estimate | Standard Error | DF | t Value | Pr > \|t\| | Alpha |
|---|---|---|---|---|---|---|---|---|
| trt | 1 | 2 | -0.8955 | 0.9285 | 3.13 | -0.96 | 0.4032 | 0.05 |
| trt | 1 | 3 | -4.1598 | 0.9285 | 3.13 | -4.48 | 0.0189 | 0.05 |
| trt | 1 | 4 | -6.8552 | 0.9285 | 3.13 | -7.38 | 0.0044 | 0.05 |
| trt | 2 | 3 | -3.2643 | 0.9285 | 3.13 | -3.52 | 0.0364 | 0.05 |
| trt | 2 | 4 | -5.9597 | 0.9285 | 3.13 | -6.42 | 0.0067 | 0.05 |
| trt | 3 | 4 | -2.6954 | 0.9285 | 3.13 | -2.90 | 0.0591 | 0.05 |

Differences of Least Squares Means

| Effect | trt | _trt | Lower | Upper |
|---|---|---|---|---|
| trt | 1 | 2 | -3.7799 | 1.9889 |
| trt | 1 | 3 | -7.0443 | -1.2754 |
| trt | 1 | 4 | -9.7396 | -3.9708 |
| trt | 2 | 3 | -6.1488 | -0.3799 |
| trt | 2 | 4 | -8.8442 | -3.0753 |
| trt | 3 | 4 | -5.5798 | 0.1890 |

There are differences between this combined inter- and intra-block treatment comparison from the intra-block treatment comparisons and are presented in the following table:

| | **Intra-block** | | | **Combined Intra- and Inter-block** | | |
|---|---|---|---|---|---|---|
| **Treat** | **Estimate** | **Lower** | **Upper** | **Estimate** | **Lower** | **Upper** |
| 1 vs 2 | -0.850000 | -3.802709 | 2.102709 | -0.8955 | -3.7799 | 1.9889 |
| 1 vs 3 | -4.125000 | -7.077709 | -1.172291 | -4.1598 | -7.0443 | -1.2754 |
| 1 vs 4 | -6.825000 | -9.777709 | -3.872291 | -6.8552 | -9.7396 | -3.9708 |
| 2 vs 3 | -3.275000 | -6.227709 | -0.322291 | -3.2643 | -6.1488 | -0.3799 |
| 2 vs 4 | -5.975000 | -8.927709 | -3.022291 | -5.9597 | -8.8442 | -3.0753 |
| 3 vs 4 | -2.700000 | -5.652709 | 0.252709 | -2.6954 | -5.5798 | 0.1890 |

Note that the confidence intervals in the combined intra- and inter-block analysis are slightly narrower than the confidence intervals in the intra-block analysis. So adding the inter-block information increases the precision for detecting differences. This is not always the case. Just as we saw with the crossover study, whether or not the block should be treated as fixed or random is dependent on the ratio of the intra-block variation to the residual variation. In our example here, that ratio was large (17.85/.85). If this ratio had been small, then the increase in precision that we saw may even become a decrease in precision.