Introduction

Matrix Algebra

Definitions

A **matrix** is a rectangular array of numbers with n rows and m columns

$$A_{3x5} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

A **vector** is a column of number or an n rows by 1 column matrix

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

A **scalar** is a single number.

A **square matrix** is a matrix with the number of rows (n) equal to the number of columns (m). In this case, the elements a_{11}, a_{22}, a_{33}, a_{44}, ..., a_{nn},are called the **diagonal elements** of the matrix and the sum of the diagonal elements is called the **trace** of the matrix.

$$A_{3x3} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 19 \end{bmatrix}$$ The trace of A=3+5+19=27.

An **identity matrix** is a square matrix with all 1's on the diagonal.

$$I_{3x3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A **transpose** of a matrix is the matrix whose columns are its rows and is indicated with a prime (`).

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 3 \\ 5 \end{bmatrix} \qquad \mathbf{a'} = \begin{bmatrix} 1 & 2 & 4 & 8 & 3 & 5 \end{bmatrix}$$

$$A_{3x3} = \begin{bmatrix} 3 & 4 & 8 \\ 2 & 4 & 9 \\ 1 & 5 & 6 \end{bmatrix} \qquad A'_{3x3} = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 4 & 5 \\ 8 & 9 & 6 \end{bmatrix}$$

Rows (or columns) are **linear dependent** if a row is a linear combination of one or more other rows in the matrix.

$$A_{3x3} = \begin{bmatrix} 3 & 4 & 14 \\ 2 & 4 & 12 \\ 1 & 5 & 12 \end{bmatrix}$$ 2*Column 1 + 2*Column 2=Column 3.

The **rank** of any matrix is the number of **linearly independent** rows (or columns) of the matrix.

Properties of rank

1. Rank is always positive.
2. For a rectangular matrix, the rank is always $\leq$ the smaller of rows or columns.
3. For a square matrix, the rank is always $\leq$ its order.

4. If the rank of a square matrix < its order, then its inverse does not exist. (More on inverses later)

Matrix Addition and Subtraction

The sum of two matrices is the matrix of sums, element by element

$$A_{3x3} = \begin{bmatrix} 3 & 4 & 8 \\ 2 & 4 & 9 \\ 1 & 5 & 6 \end{bmatrix}$$

$$B_{3x3} = \begin{bmatrix} 2 & 8 & 8 \\ 7 & 5 & 1 \\ 7 & 4 & 4 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5 & 12 & 16 \\ 9 & 9 & 10 \\ 8 & 9 & 10 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & -4 & 0 \\ -5 & -1 & 8 \\ -6 & 1 & 2 \end{bmatrix}$$

Matrices must have the same order in order to be conformable for addition or subtraction.

Scalar Multiplication

Given a scalar a=3 and the matrix A_{3x3} defined above

$$a\begin{bmatrix} 3 & 4 & 8 \\ 2 & 4 & 9 \\ 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 24 \\ 6 & 12 & 27 \\ 3 & 15 & 18 \end{bmatrix}$$

Thus the matrix A multiplied by the scalar *a* is the matrix A with every element multiplied by *a*.

Matrix Multiplication

$$A_{4x3} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 3 & 2 \end{bmatrix}$$

$$B_{3x2} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$AB_{(4x3)(3x2)} = \begin{bmatrix} (1)(1)+(0)(0)+(2)(0)=1 & (1)(2)+(0)(1)+(2)(-1)=0 \\ (3)(1)+(1)(0)+(1)(0)=3 & (3)(2)+(1)(1)+(1)(-1)=6 \\ (1)(1)+(2)(0)+(1)(0)=1 & (1)(2)+(2)(1)+(1)(-1)=3 \\ (-1)(1)+(3)(0)+(2)(0)=-1 & (-1)(2)+(3)(1)+(2)(-1)=-1 \end{bmatrix}_{4x2}$$

The jth column of B must have the same number of elements as the ith row of A in order for the matrices to be conformable for multiplication.

Given A and c, we can solve for b with the following **system of equations**

$Ab = c \quad => \quad b=A^{-1}c$

The **inverse of a matrix** is the matrix whose product with the original matrix is the identity matrix.

$$A_{3x3} = \begin{bmatrix} 3 & 4 & 8 \\ 2 & 4 & 9 \\ 1 & 5 & 6 \end{bmatrix}$$

$$A^{-1}{}_{3x3} = \begin{bmatrix} 0.7777778 & -.592593 & -.148148 \\ 0.11111 & -.37037 & 0.4074074 \\ -0.222222 & 0.4074074 & -0.148148 \end{bmatrix}$$

If a matrix is **full rank** then its inverse exists.

If the matrix is not full rank, it is called **singular** its inverse does not exist.

Provided Ab=c has a solution, then a solution is b= A^{-}c

A **generalized inverse** of *nxm* matrix A, denoted by A^{-}, is any *mxn* matrix that satisifies the following relationship

$AA^{-}A=A$

Note that there are an infinite number of generalized inverses.

Example:

$$A = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 3 & 3 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad b^0 = \begin{bmatrix} \mu^0 \\ \alpha_1^0 \\ \alpha_2^0 \\ \alpha_3^0 \end{bmatrix} \quad c = \begin{bmatrix} 96 \\ 45 \\ 24 \\ 27 \end{bmatrix}$$

	b_1^0	b_2^0	b_3^0	b_4^0
μ_0^0	16	14	27	-2982
α_1^0	-1	1	-12	2997
α_2^0	-4	-2	-15	2994
α_3^0	11	13	0	3009

The four b^0 presented above are four of an infinite number of solutions for the parameters. However, an investigator is normally not interested in the treatment solutions, but in specific expressions that can be described with a **linear functions** of the solutions of b^0 including the following:

α_1^0 - α_2^0 - difference of effects of two treatment levels = 3 for all four b^0
μ_0^0 + α_1^0 - general mean plus effect of the first treatment level = 15
½(α_2^0 + α_3^0) - α_1^0 - superiority of mean effect of levels 2 and 3 over effect of level 1 = 5.5

These linear functions are **invariant** to whatever solution is obtained.