Linear Systems and Matrix Algebra Notes

Examples

Example 1: A bank wishes to invest a $100,000 trust fund in three sources: bonds paying 8%; certificates of deposit paying 7%, and first mortgages paying 10%. The bank wishes to realize an $8000 annual income from the investment. A condition of the trust is that the total amount invested in bonds and certificates of deposit must be triple the amount invested in mortgages. How much should the bank invest in each possible category? Let x, y, and z, respectively, be the amounts invested in bonds, certificates of deposit, and first mortgages. Solve the system of equation by the Gaussian elimination method.

From algebra we have seen how tables can be helpful in organizing information, so for a lot of these problems in chapters 2, 3, and 4 I like to use tables. Here is my table for this problem:

Investment	Quantity	Interest rate	Interest earned
Bonds	X	.08	.08x
C.D.'s	Y	.07	.07y
Mortgage	Z	.10	.10z
Available	100,000	---	8,000

I like to have my rows be the things that I am keeping track of, and the columns the information that I want about those things. Often what we want to find is the quantity or amount of something, so these are the unknowns or variables. Typically in tables the columns help us build other columns, like in our case where the product of the interest rate and the quantity produces the interest earned for each account. Notice that in the last row we keep track of what's available or desired for each of our column concepts; however, there is nothing in the interest rate column (there is no interest rate for all of the accounts mixed together) so we better be given the interest earned for the mix of all accounts, and fortunately we are. Once the information is placed how do we use the table to generate the mathematical model for the problem? In our case, the model is going to be a system of linear equations, and typically the equations come from the columns (at least in the way that I set up my tables). I also like to think of an equation as "saying the same thing twice, but in two different ways," so the trick is to find a concept worth writing about twice. In the quantity column we have the total amount to be invested, $100,000 (I love how the authors throw around money in these books!). How can I say that in another way? Well, I see that x, y, and z represent the amounts that are invested in the separate accounts, so if I add them together I should get the total. Here the sum of the parts is the whole thing (rather than greater), so there is one of my equations. Another comes from the interest earned column. Again the sum of the parts must add up to be the whole thing, which in this case is $8,000.

Here is our system so far:
$$\begin{aligned} x + y + z &= 100{,}000 \\ .08x + .07y + .10z &= 8{,}000 \end{aligned}$$

Now, you may have noticed that in order to have a good chance at obtaining a unique solution to a system it helps to have the same number of equations as variables (that is the system is square). This means that I suspect another equation around this problem somewhere, and sure enough we have some info not in the table for this one. We are told, "a condition of the trust is that the total amount invested in bonds and certificates of deposit must be triple the amount invested in mortgages." These kinds of sentences can be some of the toughest to model as an equation. The amount invested in bonds is represented as x, and the amount in CD's is y, so the amount invested in bonds and CD's is simply the sum of the two, x + y. Now we are comparing this to the amount invested in mortgages, z. From the words I get the sense that the quantity represented by x + y is a larger amount than that represented by z, so if I am going to write an equation relating the two concepts, then I need to either make the bigger amount smaller to get my equation, or I am going to have to make the small thing bigger. I chose to make the small thing (in this case z) bigger. So our last equation is x + y = 3z.

Here is our final system:
$$\begin{aligned} x + y + z &= 100{,}000 \\ .08x + .07y + .10z &= 8{,}000 \\ x + y &= 3z \end{aligned}$$

To solve this system using Gaussian elimination or sometimes referred to as Gauss-Jordan elimination, we first need to rewrite our system so that all variables are on the left of the equals and constants on the right:

$x + y + z = 100{,}000$

$.08x + .07y + .10z = 8{,}000$

$x + y - 3z = 0$

Next we rewrite this system of linear equations as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100{,}000 \\ .08 & .07 & .10 & 8{,}000 \\ 1 & 1 & -3 & 0 \end{array}\right]$$

The first pivot element is in row one and column one. To perform a pivot we need to make the pivot element a one, and then make all other entries in that column zeros. Since the first row, first column element is already a one, we need only to zero out the .08 and the 1 below it. To do this we multiply the row with the pivot element (row one here) by the opposite of what we want to make into a zero, then add the two rows. This sum then replaces the row where we want the zero. For example, to zero out the .08 we can multiply row one by -.08 (the opposite of what we want to zero), then add this to row two, and then replace row two with this sum. Here it is in a more compact notational form: $-.08R_1 + R_2 \to R_2$. Here is my work below:

$$\begin{array}{cccc} -.08 & -.08 & -.08 & -8000 \\ .08 & .07 & .10 & 8000 \\ \hline 0 & -.01 & .02 & 0 \end{array}$$

Now the last row with the sum becomes the new row two in our next equivalent matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100,000 \\ 0 & -.01 & .02 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right]$$

To zero out the one at the bottom of column one, we can perform the following row operation: $-R_1 + R_3 \to R_3$.

Here it is as scratch work:

$$\begin{array}{cccc} -1 & -1 & -1 & -100000 \\ 1 & 1 & -3 & 0 \\ \hline 0 & 0 & -4 & -100000 \end{array}$$

This sum becomes the new row three in the next equivalent matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100,000 \\ 0 & -.01 & .02 & 0 \\ 0 & 0 & -4 & -100000 \end{array}\right]$$

Column one is now pivoted, so we move to the next column (column two) where we find the next pivot element by going down two entries. This entry is not a one, so we need to multiply the row by the reciprocal of the entry: $-\frac{1}{.01}R_2 \to R_2$. Here is the matrix after performing that row operation:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100,000 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -4 & -100000 \end{array}\right]$$

The only element that needs to be zeroed out is the 1 in row one, column 2. To do this we can perform the row operation: $-R_2 + R_1 \to R_1$. Here is that done as scratch work:

$$\begin{array}{cccc} 0 & -1 & 2 & 0 \\ 1 & 1 & 1 & 100{,}000 \\ \hline 1 & 0 & 3 & 100{,}000 \end{array}$$

The sum becomes the new row one in our next equivalent matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 100{,}000 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -4 & -100000 \end{array}\right]$$

Columns one and two are now looking pretty spiffy; so we move to column 3 where we take the third element down to make our pivot. It is not a one, so we multiply row three by $-\frac{1}{4}$, to make it a one. Here is our matrix after that row operation.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 100{,}000 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 25{,}000 \end{array}\right]$$

Next, we want to zero out the 3 and the -2 in column three. Here are the row operations to do that task: $\begin{array}{l} -3R_3 + R_1 \to R_1 \\ 2R_3 + R_2 \to R_2 \end{array}$

After performing these row operations you should get the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 25{,}000 \\ 0 & 1 & 1 & 50{,}000 \\ 0 & 0 & 1 & 25{,}000 \end{array}\right]$$

This matrix is finished for it has the row reduced diagonal form (except I see that the 1 in row two, column 3 should be a 0). From this matrix we read that x = 25,000, y = 50,000 and z = 25,000. In the context of our problem this means that the bank should invest $25,000 in bonds; $50,000 in certificates of deposit, and $25,000 in first mortgages.

Example 2: A baked potato smothered with cheddar cheese weighs 180 grams and contains 10.5 grams of protein. If cheddar cheese contains 25% protein and a baked potato contains 2% protein, how many grams of cheddar cheese are there?

x = grams of cheddar cheese

180 − x = grams of potato

$.25x + .02(180 - x) = 10.5$

Two ways to express grams of protein:

protein from potato + protein from cheddar cheese and as 10.5

$.02(180 - x) + .25x = 10.5$

$2(180 - x) + 25x = 1050$

$360 - 2x + 25x = 1050$

$360 + 23x = 1050$

$23x = 690$

x = 30 grams of cheddar cheese

$$\begin{array}{r} 1050 \\ -360 \\ \hline 690 \end{array}$$

$$23 \overline{)690} = 30$$
69
00

Example 3: In a laboratory experiment, a researcher wants to provide a rabbit with exactly 1000 units of vitamin A, exactly 1600 units of vitamin C, and exactly 2400 units of vitamin E. The rabbit is fed a mixture of three foods. Each gram of food 1 contains 2 units of vitamin A, 3 units of vitamin C, and 5 units of vitamin E. Each gram of food 2 contains 4 units of vitamin A, 7 units of vitamin C, and 9 units of vitamin E. Each gram of food 3 contains 6 units of vitamin A, 10 units of vitamin C, and 14 units of vitamin E. How many grams of each food should the rabbit be fed?

Here is the table that I set up for this problem:

Food	Quantity	Vitamin A	Vitamin C	Vitamin E
I	X	2	3	5
II	Y	4	7	9
III	Z	6	10	14
Available	---	1000	1600	2400

The equations for the system come from the columns regarding the vitamin requirements. As before we need to multiply the quantity of food by the amount of vitamin per unit of food to produce a meaningful concept that we can turn into an equation. For example, for each unit of food I there are 2 units of vitamin A, so if I multiply the amount of food I, x, by 2 I get a meaningful expression 2x that represents the amount of vitamin A that is in all of the food I that is being served. Likewise 4y does the same thing for food II, and 6z represents the same concept for food III. If we add up all of these amounts of vitamin A contained in all of foods I, II, and III, we have the total amount of vitamin A being served. To make an equation we have to say the same thing twice but in a different way, so how else can we say the total amount of vitamin A? We have that number given to us at the bottom of the column, 1000 units, so we have an equation from the vitamin A column: 2x + 4y + 6z = 1000. In a like manner you can construct the other two equations in the system from the vitamin C and E requirements. Here is the system for this problem:

$$2x + 4y + 6z = 1000$$
$$3x + 7y + 10z = 1600$$
$$5x + 9y + 14z = 2400$$

What is different is that when you solve this system you find that it has an infinite number of solutions (the system ends up being dependent). As the text states, for these systems we try to get them diagonalized as best we can. Below is the sequence of pivots that I made to accomplish this:

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 1000 \\ 3 & 7 & 10 & 1600 \\ 5 & 9 & 14 & 2400 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 500 \\ 0 & 1 & 1 & 100 \\ 0 & -1 & -1 & -100 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 300 \\ 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Like the text states, we move left to right with our pivoting using the 3 rules to transform each matrix into another, but equivalent matrix where the sequence of matrices continues to get simpler and simpler to read the solution. The simplest matrix to read is the diagonalized matrix. If you cannot get your matrix completely diagonalized, then get as many columns as possible in the proper form (again doing so in a left to right fashion). In that process if you run into a matrix like the one above that has a row of zeros in it, then you have a system that is dependent (which basically means that the system contains redundancies, some of the equations are not necessary, the information they contain is embedded in the other equations of the system).

If we reconstitute the final matrix above back into a system it would look like this:

$x + z = 300$

$y + z = 100$

$0x + 0y + 0z = 0$

The last equation is not needed I simply put it in there to show that it is nothing more than an identity (a true statement for all real numbers).

Notice that the top two equations have a z. We could solve those two equations for x and y in terms of z to get the following look:

$x = 300 - z$

$y = 100 - z$

Our system reduced to an equivalent system that allows z to take on any real number value, but once it is selected, then x and y are determined by that selection. Since you could use an infinite number of choices for z, there are an infinite number of solutions to the system. However, in an application there may be physical limitations on the values. For example here, x, y and z cannot be negative quantities. That would make no sense in the context of the problem. What is a negative amount of food? So we can determine so limits on our choice of z based upon the need to keep all of our variables nonnegative. Of course z itself must be greater than or equal to zero, and from the two equations above we can determine that z cannot get any bigger than 100 or else y becomes negative, so z has the following restrictions, $0 \le z \le 100$.

To get some examples of the infinite number of solutions possible, simply choose some values for z, and then find x and y by using the equations in the reduced system. For example we could let z be 50 grams of food III, then we need x = 300 – 50 = 250 grams of food I, and y = 100 – 50 = 50 grams of food II in order to meet all of the requirements of the problem.

Example 4: Granny's Custom Quilts receives an order for a patchwork quilt made from square patches of three types: solid green, solid blue, and floral. The quilt is to be 8 squares by 12 squares, and there must be 15 times as many solid squares as floral squares. If Granny's charges $3 per solid square and $5 per floral square, and if the customer wishes to spend exactly $300, how many of each type of square may be used in the quilt?

This example is similar to the example above. Here is my table, the system of linear equations, and the sequence of matrices using Gaussian elimination.

Type of square	Quantity	Cost/square
Solid green	X	3
Solid blue	Y	3
Floral	Z	5
Available	96	300

$$x + y + z = 96$$
$$3x + 3y + 5z = 300$$
$$x + y = 15z$$

The last equation I had to conceptualize in the words of the problem, I asked myself what is the bigger quantity, x + y or z? I determined that x + y represented the larger quantity, so to make an equation I had to make the smaller z quantity larger in order to make it equal to x + y. That is why I multiplied the z by the 15.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 96 \\ 3 & 3 & 5 & 300 \\ 1 & 1 & -15 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 96 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & -16 & -96 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 96 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 90 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

Bringing this last matrix back into the form of a system I get:

$x + y = 90$

$z = 6$

So I know that we need 6 floral squares, but what about the solids? Like the previous problem we have a row of zeros, so that tips off that we have a dependent system, and that in this case it has an infinite number of solutions. So which variable do I allow to take on any real number? Our author suggests any variables whose columns are not in proper form, so in this case that would be y (the second column in the last matrix does not have a one in the second row with zeros everywhere else in the column like it should). In actuality we could also choose x to be the variable that can be any real number. Anyway, let's go with y, and then solve the first equation for x in terms of y. This yields the system:

$x = 90 - y$

$z = 6$

Where y is allowed to be any real number, but again this is in the context of a word problem, so there are restrictions imposed by the physicality of the problem. Like before it is meaningless to allow any of the variables to be negative, so y must be at least as larger a zero, but it cannot be any larger than 90 or else the number of green solid squares goes negative. Therefore we must restrict y to be: $0 \le y \le 90$.

Example 5: For what value(s) Of k will the following systems of linear equations have no solution? Infinitely many solutions?

$2x - 3y = 4$

$-6x + 9y = k$

I took the system given in problem 33 and made it into the following augmented matrix: $\left[\begin{array}{cc|c} 2 & -3 & 4 \\ -6 & 9 & k \end{array}\right]$, then I performed the pivots (making the appropriate matrix entry one and everything else in the column zero).

$$\left[\begin{array}{cc|c} 2 & -3 & 4 \\ -6 & 9 & k \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 2 \\ -6 & 9 & k \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 2 \\ 0 & 0 & k+12 \end{array}\right]$$

To make the system have an infinite number of solutions we the last row to be nothing but zeros, so that means that k must be -12 to satisfy that condition. To make the system have no solution (be inconsistent) we need the last row to have zeros on the left of the vertical line and a nonzero value to the right, so to satisfy that condition we need k to be any real number that is not -12.

Example 6: Statistics show that at a certain university, 70% of the students who live on campus during a given semester will remain on campus the following semester, and 90% of students living off campus during a given semester will remain off campus the following semester. Let x and y denote the number of students who live on and off campus this semester, and let u and v be the corresponding numbers for he next semester. Then

$$.7x + .1y = u$$
$$.3x + .9y = v$$

(a) Write this system of equations in matrix form.

(b) Solve the resulting matrix equation for $\begin{bmatrix} x \\ y \end{bmatrix}$.

(c) Suppose that out of a group of 9000 students, 6000 currently live on campus and 3000 live off campus. How many lived on campus last semester? How many will live off campus next semester?

For this problem we are given the system of equations that models the problem: $\begin{matrix} .7x + .1y = u \\ .3x + .9y = v \end{matrix}$, all we have to do is to transform it into a matrix equation, and then solve it using the inverse method. The system as a matrix equation looks like this: $\begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

Using matrix algebra (a different algebra from the one we are use to with real numbers) we can solve matrix equations by multiplying both sides of the equation by the inverse of the coefficient matrix, so we next need to determine that inverse. Since this is a 2x2 matrix the short cut is the easy route. You could make the augmented matrix and perform the steps of Gaussian elimination, but that is more work here (we have to use Gaussian elimination for larger matrices). To find the inverse with the short cut for 2x2 matrices we must determine the determinant, (.7)(.9) - (.3)(.1) = .6. Then we multiply the reciprocal of the determinant by the 2x2 matrix $\begin{bmatrix} .9 & -.1 \\ -.3 & .7 \end{bmatrix}$, which is constructed by swapping the main diagonal elements of the coefficient matrix (the .7 and the .9, and taking the opposite values of the other diagonal (no swapping along that diagonal). So now we have our inverse matrix for the coefficient matrix. We can now set up the solution in terms of this matrix: $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{.6}\begin{bmatrix} .9 & -.1 \\ -.3 & .7 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$.

In part c we are given some numbers of students living on and off campus during a certain semester, and asked to find the numbers of on and off campus students for the semester before and the semester after. To find the numbers for the previous semester we use the matrix equation $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{.6}\begin{bmatrix} .9 & -.1 \\ -.3 & .7 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$, and to find the numbers for the next semester we use the matrix equation $\begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$. So the inverse matrix brings us backwards while the original pushes us forward. Below are my computations for the numbers given in the problem for the current semester.

Last semester: $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{.6}\begin{bmatrix} .9 & -.1 \\ -.3 & .7 \end{bmatrix}\begin{bmatrix} 6000 \\ 3000 \end{bmatrix} = \begin{bmatrix} 8500 \\ 500 \end{bmatrix}$, which translates to 8500 students on campus last semester, and 500 students off campus.

Next semester: $\begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix}\begin{bmatrix} 6000 \\ 3000 \end{bmatrix} = \begin{bmatrix} 4500 \\ 4500 \end{bmatrix}$, which translates into 4500 students on campus next semester, and 4500 students off campus.

Notice something intriguing that adds a little more meaning to some of the matrix operations earlier in the chapter. Let's look at using this model to project the numbers of students on and off campus two semesters ahead. To do this we simply apply our model to the last set of figures that we evaluated, the 4500 students for both on and off campus.

Two semesters ahead: $\begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix}\begin{bmatrix} 4500 \\ 4500 \end{bmatrix} = \begin{bmatrix} 3600 \\ 5400 \end{bmatrix}$

Now take the original coefficient matrix $\begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix}$ and square it (that is multiply it by itself), you should get $\begin{bmatrix} .52 & .16 \\ .48 & .84 \end{bmatrix}$. Now multiply the original student numbers of 6000 on campus students and 3000 off campus students by this squared matrix: $\begin{bmatrix} .52 & .16 \\ .48 & .84 \end{bmatrix}\begin{bmatrix} 6000 \\ 3000 \end{bmatrix} = \begin{bmatrix} 3600 \\ 5400 \end{bmatrix}$, which amazingly matches the number of students on and off campus that we got above for two semesters away, so squaring the coefficient matrix yields a matrix that embodies the way that the student population changes over <u>two</u> semesters! If you cubed the original coefficient matrix you would find that it embodies the way that the student population changes over three semesters, and so on and so forth. Pretty cool I thought. This is a concept that some authors discuss, but others don't. I like it because it gives us another example of how matrix operations can embody meaning in the context of word problems.

Example 7: There are two age groups for a particular species of organism. Group I consists of all organisms aged under 1 year, while group II consists of all organisms aged from 1 to 2 years. No organism survives more than 2 years. The average number of offspring per year born to each member of group I is 1, while the average number of organisms per year born to each member of group II is 2. Nine-tenths of group I survive to enter group II each year.

(a) Let x and y represent the initial number of organisms in groups I and II, respectively. Let a and b represent the number of organisms in groups I and II, respectively, after one year. Write a matrix equation relating $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} a \\ b \end{bmatrix}$.

(b) If there are initially 450,000 organisms in group I and 360,000 organisms in group II, calculate the number of organisms in each of the groups after 1 year and after 2 years.

(c) Suppose that at a certain time there were 810,000 organisms in group I and 630,000 organisms in group II. Determine the population of each group 1 year earlier.

This problem is very similar to example 6, but it makes us do the system setup instead of giving us that information. I find these problems difficult to make into a matrix right off the bat, for the matrix I invariably come up with is not the one that is needed, so what I do now is take the time to make the system of linear equations, and then transform this into the matrix equations that I need. In this case we need equations that give us the next groups population in terms of the initial population numbers. Group one will have numbers for the next year based on the birth rates of the two groups, so I figure that the population for group one can be modeled as follows: a = 1x + 2y. One birth per group one member, so 1x models the idea that there will be on average one new member of group one for every current member of group one. The 2y represents the number of births into group one from group two members, so the sum of 1x and 2y represent the total number of births into group one next year, and that is what “a” represents too; thus, the equation a = 1x + 2y. For the population size of group two we have all current members of group two die, so 0y represents the number of current group two members going on to group two next year. Likewise .9x represents the number of group one members that move on to become group two members the next year; thus the equation relating the current population of group two to the next year’s population of group two is b = .9x + 0y, or more simply, b = .9x. So here is our system: $\begin{matrix} a = x + 2y \\ b = .9x \end{matrix}$. Transform this into matrix equation form and we get: $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ .9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. You can use this matrix equation to compute part b, and you can solve this equation for x and y to get the matrix equation (involving the inverse matrix) for finding past populations which is the subject of part c.

Here is a link to more examples of pivoting a matrix: Pivoting.

Return to Top of Page