I. MARKOV PROCESSES

I.1. **How to show a Markov Process reaches equilibrium.**

(1) Write down the transition matrix $P = [p_{ij}]$, using the given data.

(2) Determine whether or not the transition matrix is regular. If the transition matrix is regular, then you know that the Markov process will reach equilibrium.

Note: If the transition matrix is not regular, the Markov process still may reach equilibrium. For example, every transition matrix in problem 6 of the July 8^{th} handout reaches equilibrium. However, trying to show that a process will *not* achieve equilibrium can be a *very* tricky question, so I will not be asking this of you.

I.2. **How to find the steady-state vector of a Markov Process.**

(1) Write down the transition matrix $P = [p_{ij}]$, using the given data.

(2) Find $I_n - P$.

(3) Solve the homogeneous system $(I_n - P)\bar{\mathbf{x}} = \mathbf{0}$ as you would solve any homogeneous system:

(a) Put $(I_n - P)$ into row-echelon or reduced row-echelon form.

(b) Write this new (equivalent) matrix as a system of equations to obtain a parameterized solution set.

(c) For a solution set $(a_1t, a_2t, \ldots, a_nt)$, $t \in \mathbb{R}$, solve the equation $a_1t + a_2t + \ldots + a_nt = 1$ for t.

(d) We have now found the steady-state vector. Using the value we just found for t, write the steady-state vector as

$$\bar{\mathbf{x}} = \begin{bmatrix} a_1t \\ a_2t \\ \vdots \\ a_nt \end{bmatrix}$$

(4) The steady-state vector $\bar{\mathbf{x}}$ gives percentages; it describes the portions of the population that will eventually come to be in the various states of the system. If the problem asks for the number of members of the population that are in a particular state (i.e., if they give you an initial state vector), then you need to multiply the percents given by $\bar{\mathbf{x}}$ against the *total population* in order to find the final amount. This question is not always asked.

Example 1. If you find the steady-state vector of a Markov process to be

$$\bar{\mathbf{x}} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

and the problem gives you some data about the initial population like

"The experiment begins with 200 sick mice and 300 healthy mice."

Then compute

$$\begin{bmatrix} 0.7 \times 500 \\ 0.3 \times 500 \end{bmatrix} = \begin{bmatrix} 350 \\ 150 \end{bmatrix} \begin{matrix} \text{sick} \\ \text{healthy} \end{matrix}$$

to discover that in the long run, there will be 350 sick mice and 150 healthy mice.

Note: if the initial data (the initial state vector) is not given, then this step is not necessary - just use the percentage breakdown given by $\bar{\mathbf{x}}$ for your final solution.

Another note: the initial numbers 200 and 300 are not used in the computation of the final state numbers! We only used the total. This is part of the point of Markov processes, and one of the key points of our theorem on Markov processes (see the July 9 lecture notes). Remember, it doesn't matter what the initial state of the system is, if the transition matrix is regular, then the population will always tend to $\bar{\mathbf{x}}$.

II. Leontief Closed Models

II.1. **How to "solve" a Leontief Closed model.**

This section describes how to obtain fair prices for a the different goods in a closed economic system. This method can also be used to determine the income of the different agents involved in a closed economic system.

(1) Write down the exchange matrix $E = [e_{ij}]$. ("m_i uses e_{ij} of g_j").
(2) Find $(I_n - E)$.
(3) Solve the homogeneous system $(I_n - E)\bar{\mathbf{x}} = \mathbf{0}$ as you would solve any homogeneous system:
   (a) Put $(I_n - E)$ into row-echelon or reduced row-echelon form.
   (b) Write this new (equivalent) matrix as a system of equations to obtain a parameterized solution set $(a_1t, a_2t, \ldots, a_nt),\ t \in \mathbb{R}$.

Now you know that a fair assignment of prices will occur in the ratio

$$\$a_1 : \$a_2 : \ldots : \$a_n$$

Example 2. So if you end up with a solution like 4 : 3 : 4, then if one unit of g_1 (that is, product #1) is worth \$8, then you know that one unit of g_2 will be worth \$6.

For a problem like #4 on the Leontief Closed Model handout, suppose you end up with a solution like

$$(4t, 2t, 6t),\ t \in \mathbb{R}$$

Then one way to phrase the solution would be:

"The income of the second country is half that of the first, and the income of the third country is the sum of the other two."

But you can also just put

"The incomes of the three countries are in the ratio $2 : 1 : 3$."

Note that this technique is essentially the same as the technique for finding the steady-state vector of a Markov process, except that you don't need to solve $a_1t + a_2t + \ldots + a_nt = 1$ to find a specific t. The ratio *is* the solution.

For another example of this, see the Canada/Mexico/USA example we did on Thursday. (In my notes, this is actually the last page of the July 10th lecture.

III. Leontief Open Models

III.1. How to "solve" a Leontief Open model.

This section describes how to determine what the optimal production vector should be for a given system, i.e., how much of each good should be produced to meet the demand exactly.

(1) Write down the consumption matrix $C = [c_{ij}]$. ("c_{ij} is the dollar amount of g_i consumed in the production of \$1 worth of g_j").

(2) (a) Find $(I_n - C)$.
(b) Find $(I_n - C)^{-1}$.

(3) Use $(I_n - C)^{-1}$ to solve the matrix equation $(I_n - C)\bar{\mathbf{x}} = \bar{\mathbf{d}}$ for $\bar{\mathbf{x}}$ as follows:

$$\bar{\mathbf{x}} = (I_n - C)^{-1}\bar{\mathbf{d}}$$

Now $\bar{\mathbf{x}}$ is the optimal production vector:

$$\text{to meet a demand of } \bar{\mathbf{d}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}, \text{ you find } \bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

So that you know to produce \$$x_1$ of good g_1, \$$x_2$ of good g_2, etc, in order to meet the demand exactly.

Example 3. See the coal and steel example (exercise #4 from the July 11th homework), as worked out in the Selected Homework Solutions (available on the web site).

III.2. How to determine if the consumption matrix is productive.

The consumption matrix of a Leontief Open model is productive if $(I_n - C)^{-1}$ exists and has all nonnegative entries.

IV. A note on probability

This may be applicable to any of the applications we have discussed.

"All" or "always": In a word problem, "all" or "always" corresponds to the probability 1. An example of this would be §2.5, #11: "Those who watch an hour or more of television on any given day always watch for less than an hour the next day." This means that the corresponding entry of the transition matrix should get a 1, representing the entire dormitory population.

"None" or "never": In a word problem, "none" or "never" corresponds to the probability 0, just like "all" or "always" corresponds to the probability 1.

"Twice as likely", etc.: Suppose you have a word problem with two options that incorporates relative probabilities or relative proportions. For example, say a rat in a study goes through door A with probability a, or door B with probability b.

- "The rat has an even chance of going through door A as door B."
  This sentence means that we have $a = b$. Since $a + b = 1$ (because they are probabilities), this gives

$$\begin{aligned} a + b = 1 &\implies a + a = 1 \\ &\implies 2a = 1 \\ &\implies a = \tfrac{1}{2} \end{aligned}$$

  Thus $a = \frac{1}{2}$ and $b = \frac{1}{2}$, and we have two equal numbers whose sum is 1.
- "The rat is twice as likely to go through door B as through door A."
  This sentence means that $b = 2a$. Since $a+b = 1$ (because they are probabilities), this gives

$$\begin{aligned} a + b = 1 &\implies a + 2a = 1 \\ &\implies 3a = 1 \\ &\implies a = \tfrac{1}{3} \end{aligned}$$

  Thus $a = \frac{1}{3}$ and $b = \frac{2}{3}$, and we have two numbers whose sum is 1, with the second being twice the first.
- "The rat is three times as likely to go through door B as through door A."
  Follow this through yourself, to see that $a = \frac{1}{4}$ and $b = \frac{3}{4}$.

Further relative proportions: Suppose you have a word problem where three options are possible. For example, say our rat can now go through door A with probability a, door B with probability b, and door C with probability c. Now suppose that the rat is twice as likely to choose B as A, and three times as likely to choose C over A. The first sentence tells you that $b = 2a$. The second sentence tells you that $c = 3a$. Since these are probabilities, we must have $a + b + c = 1$. Thus

$$\begin{aligned} a + b + c = 1 &\implies a + 2a + 3a = 1 \\ &\implies 6a = 1 \\ &\implies a = \tfrac{1}{6} \end{aligned}$$

Thus $a = \frac{1}{6}$, $b = \frac{2}{6} = \frac{1}{3}$, and $c = \frac{3}{6} = \frac{1}{2}$.