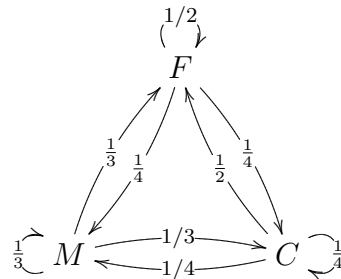


Mathematics-AppliedMatrixAlgebra

SUPPLEMENT 2

Application: Economic Models for Exchange of Goods.

Suppose that in a primitive society, the members of a tribe are engaged in three occupations: farming, manufacturing of tools and utensils, and the weaving and sewing of clothing. Assume that initially the tribe has no monetary system and that all goods and services are bartered. Let the three groups be denoted by F , M , and C , respectively. Suppose that this directed graph indicates how the bartering system works in practice:



This figure indicates that the farmers keep half of their produce and give one quarter of their produce to the manufacturers and on quarter to the clothing producers. The manufacturers divide the goods evenly among three groups, one third goes to each (including themselves). The group producing clothes gives half of the clothes too the farmers and divides the other half evenly between themselves and the manufacturers. The results are summarized in this table:

	F	M	C
F	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
M	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$
C	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$

The first column of the table indicates the distribution of the goods produced by the farmers, the second column indicates the distribution of the manufactured goods, and the third column indicates the distribution of the clothing.

As the size of the tribe grows, the system of bartering becomes too cumbersome and, consequently, the tribe decides to institute a monetary system of exchange. For this simple economic system, we assume that there will be no accumulation of capital or debt and that the prices for each of the three types of goods will reflect the values of the existing bartering system. The question is how to assign values to the three types of goods that fairly represent the current bartering system.

This problem can be turned into a linear system using a model that was originally developed by the Nobel prize winning economist Wassily Leontief, a model that we will study more extensively later. For now, let x_1 be the monetary value of the goods produced by the farmers, x_2 be the value of the manufactured goods, and x_3 be the value of the clothing produced. According to the first row of the table, the value of the goods received by the farmers amounts to half the value of the farm goods produced, plus one-third the value of

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SUPPLEMENT 1

the manufactured products, and half the value of the clothing goods. Thus the total value of the goods received by the farmers is $\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3$. If the system is fair, the total value of goods received by the farmers should equal x_1 , the total value of the farm goods produced. Thus we have the linear equation

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 = x_1.$$

Using the second row of the table and equating the value of the goods produced and received by the manufacturers, we obtain a second equation:

$$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_2.$$

Finally, the third row of the table yields:

$$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_3.$$

These equations can be rewritten as a homogeneous system:

$$\begin{aligned} -\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 &= 0 \\ \frac{1}{4}x_1 - \frac{2}{3}x_2 + \frac{1}{4}x_3 &= 0 \\ \frac{1}{4}x_1 + \frac{1}{3}x_2 - \frac{3}{4}x_3 &= 0 \end{aligned}$$

The reduced row-echelon form of the augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

We can find a parametric representation for the solution set of this system by letting $x_3 = t$. Then we immediately get $x_2 = t$ and $x_1 = \frac{5}{3}t$ by back-substitution. It follows that the variables x_1, x_2, x_3 should be assigned values in the ratio

$$x_1 : x_2 : x_3 = 5 : 3 : 3$$

Consider the parallel with modern currency systems; it doesn't matter what the actual units of money are, just that one can find a fair system of exchange.

This simple system is an example of the closed Leontief input-output model. Leontief's models are fundamental to our understanding of economic systems.

Problem 1. Determine the relative values of x_1, x_2, x_3 if the distribution of goods is as described in this table:

	<i>F</i>	<i>M</i>	<i>C</i>
<i>F</i>	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
<i>M</i>	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
<i>C</i>	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$