Mathematics-AppliedMatrixAlgebra

SUPPLEMENT 1

Application: Production Planning

A manufacturer makes three different types of chemical products: A, B, and C. Each product must go through two processing machines, X and Y. The products require the following times in machines X and Y:

(1) One ton of A requires 2 hours in machine X and 2 hours in machine Y.
(2) One ton of B requires 3 hours in machine X and 2 hours in machine Y.
(3) One ton of C requires 4 hours in machine X and 3 hours in machine Y.

Machine X is available 80 hours per week and machine Y is available 60 hours per week. Since management does not want to keep the expensive machines X and Y idle, it would like to know how many tons of each product to make so that the machines are fully utilized. It is assumed that the manufacturer can sell as much of each product as is made.

To solve this problem, let x_1, x_2, x_3 denote the number of tons of products A, B, and C, respectively, to be made. The number of hours that machine X will be used is

$$2x_1 + 3x_2 + 4x_3,$$

which must equal 80, so we get the equation

$$2x_1 + 3x_2 + 4x_3 = 80.$$

Similarly, the number of hours machine Y will be used is

$$2x_1 + 2x_2 + 3x_3 = 60.$$

Mathematically, the problem is to find a solution set for the system of equations

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 80 \\ 2x_1 + 2x_2 + 3x_3 &= 60 \end{aligned}$$

Since we have 3 variables and only 2 equations, this system will have many (in fact, infinitely many) possible solutions. We need to find a parametric representation for the solution set of this system. Suppose we let $x_3 = t$, so that t is the number of tons of C that is produced. First, notice that it is impossible (even nonsensical!) to make a negative amount of anything. This indicates that the smallest amount of C that can be made is 0. Now, since even when x_1, x_2 are both 0, x_3 cannot be over 20, it is clear that we must have $0 \leqslant t \leqslant 20$.

For $x_1 = 0$, we obtain the system

$$\begin{aligned} 3x_2 + 4t &= 80 \\ 2x_2 + 3t &= 60 \end{aligned}$$

which we can solve to get

$$x_2 = 20 - t.$$

Substituting this into one of the equations of the previous system, we obtain

$$2x_1 + 3(20 - t) + 4t = 80$$

which simplifies to

$$2x_1 + t = 20 \qquad \text{or} \qquad x_1 = \tfrac{20-t}{2}$$

Thus we see that all solutions are given by

$$x_1 = \tfrac{20-t}{2}$$
$$x_2 = 20 - t$$
$$x_3 = t$$

where $0 \leqslant t \leqslant 20$.

For example, when $t = 10$, the original system is satisfied by

$$x_1 = 5, \qquad x_2 = 10, \qquad x_3 = 10$$

and when $t = 6$, the original system is satisfied by

$$x_1 = 7, \qquad x_2 = 14, \qquad x_3 = 6.$$

Any such solution is as good as any other, and there are infinitely many of them. There is no best solution unless additional information or restrictions are given.

For each of the following problems, you should note whether the system has a solution set consisting of

- exactly one solution
- no solution
- infinitely many solutions.

Problem 1. An oil refinery produces low-sulphur and high-sulphur fuel. Each ton of low-sulfur fuel requires 5 minutes in the blending plant and 4 minutes in the refining plant; each ton of high-sulfur fuel requires 4 minutes in the blending plant and 2 minutes in the refining plant. If the blending plant is available for 3 hours and the refining plant is available for 2 hours, how many tons of each type of fuel should be manufactured so that the plants are fully utilized?

Problem 2. A dietician is preparing a meal consisting of foods A, B, and C. The nutritional information for these types of food is (in units per ounce of food):

	A	B	C
protein	2	3	3
fats	3	2	3
carbohydrates	4	1	2

If the meal should provide exactly 25 units of protein, 24 units of fat, and 21 units of carbohydrate, how many ounces of each type of food should be used?

Problem 3. An inheritance of \$24,000 is to be divided among three trusts, with the second trust receiving twice as much as the first trust. The three trusts pay interest at the rates of 9%, 10%, and 6% annually, respectively, and return a total in interest of \$2210 at the end of the first year. How much was invested in each trust?