Joint, Marginal, and Conditional Distributions

Problems involving the joint distribution of random variables X and Y use the pdf of the joint distribution, denoted $f_{X,Y}(x,y)$. This pdf is usually given, although some problems only give it up to a constant. The methods for solving problems involving joint distributions are similar to the methods for single random variables, except that we work with double integrals and 2-dimensional probability spaces instead of single integrals and 1-dimensional probability spaces. We illustrate these methods by example.

Discrete Case: Analogous to the discrete single random variable case, we have

$$0 \le f_{X,Y}(x,y) = \Pr((X = x) \wedge (Y = y)) \le 1$$

The Continuous Case is illustrated with examples.

The Mixed Case (one of the random variables is discrete, the other is continuous) is also illustrated with examples.

cdf of Joint Distribution – denoted $F_{X,Y}(x,y)$

$$F_{X,Y}(x,y) = \Pr((X \le x) \cap (Y \le y))$$

Notice that we can get the cdf's for X and Y from the joint cdf as follows:

$$F_X(x) = F_{X,Y}(x,\infty)$$
$$F_Y(y) = F_{X,Y}(\infty,y)$$

cdf of Joint Distribution (continued)

Discrete Case: We have $F_{X,Y}(x,y) = \sum_{s=-\infty}^{x}\sum_{t=-\infty}^{y} f_{X,Y}(s,t)$

Continuous Case: We have

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

General Expectation using Joint Distribution

Discrete Case: We have $E[h(X,Y)] = \sum_{x}\sum_{y} h(x,y)\cdot f_{X,Y}(x,y)$

Continuous Case: We have $E[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\cdot f_{X,Y}(x,y)\,dy\,dx$

The covariance of random variables X and Y is a generalization of variance of a single random variable

$$Cov(X,Y) = Cov(Y,X) = E[(X-\mu_X)\cdot(Y-\mu_Y)] = E[X\cdot Y] - E[X]\cdot E[Y]$$

Remarks:

1. $Var(X) = Cov(X,X)$

2. $Cov(aX+bY, cZ+d) = a\cdot c\cdot Cov(X,Z) + b\cdot c\cdot Cov(Y,Z)$ where a, b, c, and d are constants and X, Y, and Z are random variables

3. $Var(X+Y) = Cov(X+Y, X+Y) = Var(X) + Var(Y) + 2\cdot Cov(X,Y)$

The mgf for Joint Distribution is a generalization of mgf for a single random variable

$$M_{X,Y}(s,t) = E[e^{s\cdot X + t\cdot Y}]$$

Remarks:

1. $E[X^n \cdot Y^m] = \frac{\partial^{n+m}}{(\partial^n s)(\partial^m t)} M_{X,Y}(s,t)|_{(s,t)=(0,0)}$

   Important special case are

   $E[X \cdot Y] = \frac{\partial^2}{\partial s \partial t} M_{X,Y}(s,t)|_{(s,t)=(0,0)}$, $E[X] = \frac{\partial}{\partial s} M_{X,Y}(s,t)|_{(s,t)=(0,0)}$

   and $E[Y] = \frac{\partial}{\partial t} M_{X,Y}(s,t)|_{(s,t)=(0,0)}$

2. $M_X(t) = M_{X,Y}(t,0)$ and $M_Y(t) = M_{X,Y}(0,t)$

   So we also have $E[X] = \frac{d}{dt} M_{X,Y}(t,0)|_{t=0}$ and $E[Y] = \frac{d}{dt} M_{X,Y}(0,t)|_{t=0}$

Marginal Distributions (pdfs) of X and Y

Discrete Case: We have
$$f_X(x) = \sum_y f_{X,Y}(x,y)$$
$$f_Y(y) = \sum_x f_{X,Y}(x,y)$$

Continuous Case: We have
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Notice that in order to get the marginal distribution of X, we sum out (discrete case) or integrate out (continuous case) Y. Similarly, for getting the marginal distribution for Y, we sum out (discrete case) or integrate out (continuous case) X.

Conditional Distributions for *X*, given *Y* and for *Y*, given *X*

$$f_{X|Y=y}(x\,|\,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$
$$f_{Y|X=x}(y\,|\,x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

One way to remember these is by saying the words: the conditional distribution is the joint distribution divided by the marginal distribution. Also notice the probability interpretation when *X* and *Y* are discrete.

Independence of the jointly distributed random variables *X* and *Y*

If *X* and *Y* are independent, then each of the following will be true:

1. $f_{X|Y=y}(x\,|\,y) = f_X(x)$ and $f_{Y|X=x}(y\,|\,x) = f_Y(y)$
   and therefore we have $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$
   (Think of the probability interpretation in the discrete case.)

2. $E[X \cdot Y] = E[X] \cdot E[Y]$ and more generally,
   $E[h(X) \cdot g(Y)] = E[h(X)] \cdot E[g(Y)]$

3. $Cov(X, Y) = 0$

4. $Var(X + Y) = Var(X) + Var(Y)$

Double Expectation Theorem (Very Important and Useful)

$$E[X] = E[E[X\,|\,Y]]$$
$$Var(X) = E[Var(X\,|\,Y)] + Var(E[X\,|\,Y])$$