Basic and Conditional Probability

Probability Concepts – The collection of all possible outcomes when an experiment is performed is called a probability space, denoted *S*. An event is a subset of the probability space. Events are usually denoted by capital letters (*A*, *B*, etc.) Each event has a probability of occurring, which is essentially how big the event is in comparison to the experiment. The notation is that the probability of event *A* is denoted Pr(*A*). We illustrate the notation and basic probability concepts using a simple example.

Mutually Exclusive Events – Event *A* and *B* are mutually exclusive if there is no outcome common to both *A* and *B*, i.e. $A \cap B = \varnothing$.

DeMorgan's Laws – These are often tested rules that relate the set operations; union, intersection, and complements,

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

Note: DeMorgan's Laws can be generalized to more than two sets. E.g.

$$(A \cup B \cup C)' = A' \cap B' \cap C'$$
$$(A \cap B \cap C)' = A' \cup B' \cup C'$$

Other often tested set operations and probability rules:
(These rules can also be generalized)

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3. $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

4. $\Pr(A') = 1 - \Pr(A)$

5. $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$ (will be used often)

We illustrate these facts by examples.

Conditional Probability – denoted by $\Pr(B \mid A)$, is the probability that event B occurs given that event A has occurred. The formula for $\Pr(B \mid A)$ is

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

Note that we can rewrite the above formula as

$$\Pr(A \cap B) = \Pr(B \mid A) \cdot \Pr(A) = \Pr(A \mid B) \cdot \Pr(B)$$

Together with Rule 5 from above, we get the following often used result

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B') = \Pr(A \mid B) \cdot \Pr(B) + \Pr(A \mid B') \cdot \Pr(B')$$

We illustrate how to use this formula with an example.

Bayes' Rule (Theorem) – This is an often tested technique used to solve a certain type of problem. We will be asked to find $\Pr(B \mid A)$ and we do so by first using the above formula to find $\Pr(A)$. Notice that $\Pr(A)$ is a sum of terms, one of which is $\Pr(A \cap B)$. Therefore we have all the information needed to calculate $\Pr(B \mid A)$. We illustrate how to use this theorem with a couple of examples.

Independent Events – By definition, events A and B are independent events if $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$. This is equivalent to the statement that $\Pr(B \mid A) = \Pr(B)$. More generally, for any events A, B, and C¸ we have $\Pr(A \cap B \cap C) = \Pr(A \cap B \mid C) \cdot \Pr(C) = \Pr(A \mid B \cap C) \cdot \Pr(B \mid C) \cdot \Pr(C)$, whereas if A, B, and C are mutually independent then $\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$

Other conditional probability and independence rules:
(these rules can be generalized)

1. $\Pr(A \cup B \mid C) = \Pr(A \mid C) + \Pr(B \mid C) - \Pr(A \cap B \mid C)$

2. $\Pr(A' \mid B) = 1 - \Pr(A \mid B)$

We illustrate how to use these formulas with an example.