Section 6 – Short Sales, Yield Curves, Duration, Immunization, Etc.

Short Sales:

Suppose you believe that Company X's stock is overpriced. You would certainly not buy any of Company X's stock, and in fact, you would probably sell any stock that you did own in Company X. However, what if you don't currently own any stock in Company X? Is there a way for you to take advantage of Company X's stock being overpriced in your opinion? The answer is yes. The term selling short refers to an investor that sells a stock at the current price of the stock and buys the stock back at a later date to "cover the short". An investor would short sale a stock if the investor believes the stock is overpriced and thus the price will decline in the future. The following is an example that illustrates how the transaction of selling short works.

Investor A believes Company X's stock, currently priced at $10 per share, is overpriced. So A wants to go short 100 shares of Company X's stock. Then

1. A borrows 100 shares of Company X's stock from a second party and immediately sells the 100 shares in the market for $1000.
2. A will have to make a deposit in order to borrow the 100 shares above. This deposit is called a margin and is usually a percentage of the amount being borrowed. For example, let's assume the margin is 50% of the amount borrowed. Then the margin equals $500. This is the amount that A has to put up front.
3. A will earn interest on the margin. However, A will not have access to the $1000 proceeds from the short sale until A covers the short sale by buying 100 shares at a later date, thus completing the transaction.
4. If Company X's stock pays dividends, then A will have to pay the dividend amount to the second party above, since the second party lent the stock to A and thus is not receiving the dividend payments directly.
5. At the time the short sale is covered, the net profit earned by A will be the sum of the gain on the short sale and interest on the margin, offset by the amount of dividends paid by A to the second party. The yield rate on the transaction will equal the ratio of the net profit to the margin.

Yield Rates on Short Sales:

We will be interested in calculating the yield rate on a short sale. The formula for calculating the yield rate on a short sale is

$$i = \frac{G + I_M - D}{M}$$, where

M = margin
G = gain on the short sale
I_M = interest on the margin, and
D = dividends paid

The following example, which refers to the above scenario, illustrates this calculation.

Suppose after 1 year Company X's stock has declined to $8 per share. Also suppose A has earned $40 in interest on the margin and paid $60 in dividends during the year. If A decides to cover the short sale at this time, then A will buy 100 shares of Company X's stock at $8 per share. So A pays $800 but then gets the $1000 from the short sale of 1 year earlier for a gain of $200 from the short sale. Adding the interest on the margin and backing off the dividends paid by A results in a net profit of 200+40-60=180 on a margin of 500. So the yield rate on this transaction is 180/500=0.36 (36%).

Pricing Stocks: The theoretical price of a stock is calculated using the dividend discount model. This just means the price of the stock is equal to the present value of the dividends, which normally are assumed to continue forever.

Recognition of Inflation:

The following example illustrates how inflation affects the purchasing power of money. Suppose you have \$100 now and a gallon of milk currently costs \$4. Then the \$100 will buy currently buy 25 gallons of milk. Now suppose the \$100 is invested for 2 years at an 8% annual effective rate of interest and the annual rate of inflation over this 2 year period is 5%. Then at the end of the two years, you have $100(1.08)^2 = 116.64$ but milk costs $4(1.05)^2 = 4.41$ per gallon. So you can now buy 116.64/4.41 = 26.45 gallons of milk. The real rate of return, i', is measured by solving

$$25(1+i')^2 = 26.45.$$

Thus we have:

$$\begin{aligned} &25(1+i')^2 = 26.45 \\ &\Rightarrow \frac{100(1+i')^2}{4} = \frac{100(1.08)^2}{4(1.05)^2} \\ &\Rightarrow (1+i')(1.05) = 1.08 \end{aligned}$$

Let i denote the nominal rate of interest and r denote the rate of inflation. The formula relating these variables is

$$\begin{aligned} &(1+i')(1+r) = 1+i \\ &\Rightarrow 1+i' = \frac{1+i}{1+r} \end{aligned}$$

Yield Curves:

Not only are short-term and long-term interest rates generally different at any point in time, but they also change over time. This phenomenon is called the term structure of interest rates.

The following table is a hypothetical table illustrating the term structure of interest rates. We can extend the values in this table to a continuous graph, and the resulting graphical illustration is called the yield curve corresponding to table. The interest rates in the table are called spot rates.

Hypothetical Term Structure of Interest Rates

Length of Investment	Interest Rate
1 year	7.00%
2 years	8.00%
3 years	8.75%
4 years	9.25%
5 years	9.50%

The following examples illustrate how we use spot rates from a yield curve.

If person *A* invests 100 for 2 years and person *B* invests 100 for 3 years, both using the corresponding spot rates in the table above, then *A* would have $100(1.08)^2 = 116.64$ at the end of 2 years, and *B* would have $100(1.0875)^3 \doteq 128.61$ at the end of 3 years.

The following example illustrates the concept of a forward rate.

A company needs to borrow money for two years. The company can either
1) borrow money for 2 years at the spot rate in the table above, or
2) borrow money for 1 year at the 1 year spot rate in the table above, and then borrow money for the second year at the 1 year spot rate in effect 1 year from now, denoted by $f_{[1,2]}$. We call $f_{[1,2]}$ a forward rate. Find $f_{[1,2]}$ such that the company is indifferent to the two options.

We find the forward rate $f_{[1,2]}$ as follows:

$$(1.08)^2 = (1.07)(1+f_{[1,2]}) \Rightarrow f_{[1,2]} \doteq .0901 = 9.01\%$$

Duration:

The duration of a sequence of future payments is a measure of the timing of the future payments. We saw a similar idea earlier when we studied the method of equated time. We use the following example to help us recall the method of equated time.

Suppose payments of 2000, 4000, and 10000 are to be made at times 1, 2, and 4, respectively. The method of equated time produces the value

(2000*1 + 4000*2 + 10000*4) / (2000 + 4000 + 10000) = 3.125.

Since the denominator equals 16000, we can rewrite this value as the sum

[2000 / 16000]*1 + [4000 / 16000]*2 + [10000 / 16000]*4 = 3.125.

Written this way, we can think of the value produced by the method of equated time as being a weighted average of the timing of the payments. That is, the average time of the payments, taking into account the amounts of the payments, is at time 3.125. Notice that the weight given to the payment at time t equals the ratio of the *amount* of the payment at time t to the total *amount* of all payments.

The concept of duration is very similar to the concept of equated time. The calculation of duration is again a weighted average of the timing of the payments, except with duration the weight given to the payment at time t equals the ratio of the *present value* of the payment at time t to the total *present value* of all payments.

If we discount the payments above using an annual effective interest rate of 25%, then the present value of the payment made at time 1 is 1600, the present value of the payment made at time 2 is 2560, and the present value of the payment made at time 4 is 4096. The sum of these present values is 8256, and so the duration of this cash flow is

$$\bar{d} = \frac{1600}{8256} \cdot 1 + \frac{2560}{8256} \cdot 2 + \frac{4096}{8256} \cdot 4 = \frac{1 \cdot 1600 + 2 \cdot 2560 + 4 \cdot 4096}{8256} = 2.798.$$

We usually don’t discount each individual payment like in this example, but rather we use notation and formulas developed earlier.

The formula for the (Macaulay) duration of a cash flow consisting of payments of R_t at time t is

$$MacD = \frac{\sum t \cdot v^t \cdot R_t}{\sum v^t \cdot R_t}$$

Note that this value will depend upon the interest rate used for discounting. If we use a 0% interest rate in the calculation of duration, we get the same value as if we use the method of equated time.

The net present value function of a cash flow consisting of payments of R_t at time t is

$$P(i) = \sum (1+i)^{-t} R_t = \sum v^t \cdot R_t$$

Similarly to how we defined the force of interest, we define the <u>volatility</u> of a cash flow.

$$\bar{v} = -\frac{P'(i)}{P(i)}$$

The negative sign in front ensures a positive value for the volatility.

An interesting result appears if we carry out the algebra in the defining expression for $\bar{v}$.

Taking a derivative with respect to i of $P(i)$, we get

$$P'(i) = \sum -t \cdot (1+i)^{-t-1} \cdot R_t = -\sum t \cdot v^{t+1} \cdot R_t = -v \cdot \sum t \cdot v^t \cdot R_t$$

Then plugging this back in the formula $\bar{v} = -\frac{P'(i)}{P(i)}$, we get

$$\bar{v} = \frac{-P'(i)}{P(i)} = \frac{-(-v\sum t \cdot v^t \cdot R_t)}{\sum v^t \cdot R_t} = v \cdot \frac{\sum t \cdot v^t \cdot R_t}{\sum v^t \cdot R_t}$$

But the second factor in the last expression is $\frac{\sum t \cdot v^t \cdot R_t}{\sum v^t \cdot R_t} = MacD$, the (Macaulay) duration of the cash flow. So we get the following relationship between (Macaulay) duration and volatility: (For this reason, volatility is also called modified duration.)

$$\bar{v} = ModD = v \cdot MacD = \frac{MacD}{1+i}$$

Investing Assets versus Liabilities:

Analogous to the concept of volatility (modified duration) is the concept of convexity. The formula for the convexity of a cash flow is

$$\bar{c} = \frac{P''(i)}{P(i)}$$

We study three strategies of investing assets versus liabilities. (This means we will invest money now and use our return on the investment to pay future liabilities.)

Method 1: (Immunization)

Immunization is a technique to structure assets and liabilities in a manner that would reduce, or eliminate, the risk of adverse effects created by **small** changes in interest rates.

Let $R_t = A_t - L_t$ = the net receipts at time t. Immunization is achieved at interest rate i_0 if the net present value function $P(i) = \sum (1+i)^{-t} R_t = \sum v^t \cdot R_t$ has a local minimum at i_0 and $P(i_0) = 0$. Thus, immunization is achieved if

$$P(i_0) = 0$$
$$P'(i_0) = 0$$
$$P''(i_0) > 0$$

Note: An immunization strategy is to arrange assets so that

1. The present value of cash inflow from assets is equal to the present value of cash outflow from liabilities.
2. The volatility (modified duration) of the assets is equal to the volatility (modified duration) of the liabilities.
3. The convexity of the assets is greater than the convexity of the liabilities.

Method 2: (Full Immunization)

Full immunization is a technique to structure assets and liabilities in a manner that would reduce, or eliminate, the risk of adverse effects created by **all** changes in interest rates.

Suppose we have one liability outflow L_k at time k. The concept is to hold two assets, one that will produce a cash inflow, A, prior to the liability outflow (say at time $k-a$) , and another that will produce a cash inflow subsequent to the liability outflow (say at time $k+b$). We use the force of interest δ that is equivalent to i and we use the liability payment time as the comparison date. We want

$P(\delta) = Ae^{a\delta} + Be^{-b\delta} - L_k = 0$ and $P'(\delta) = Aae^{a\delta} - Bbe^{-b\delta} = 0$.

The full immunization strategy is obtained by solving this system of two equations. Repeat the above process for each liability outflow. There will be two asset inflows for each liability outflow.

Method 3: Absolute Matching (also called Dedication)

The idea here is to structure assets such that the resulting asset inflows will exactly match the liability outflow.