Module 1: Interest and Discount

Section 2: Accumulation Functions and Equivalent Rates

The most important function to master in the early part of Interest Theory is the **accumulation function**, for which we'll use $a(t)$ for the defining expression. The independent variable (input) for this function, t, is the time period, measured in the appropriate units. The output represents the accumulated value at time t of an initial investment of 1 made at time 0. The timeline is as follows:

Insert Timeline

The defining expression for the accumulation function will depend on the way interest is being credited. Actually, we've already seen this function in the last section. We have three different accumulation functions so far; namely,

$a(t) = 1 + it$ (simple interest accumulation function – t measured in years)
$a(t) = (1 + i)^t$ (compound interest accumulation function – t measured in same periods as i is in)
$a(t) = e^{\delta t}$ (*constant* force of interest accumulation function – t measured in years)

Notice these are derived from the **amount functions** from the previous section by substituting $P = 1$ in the appropriate formulas. In this manual I will generally focus on accumulation functions (having initial investment amount of 1) since if the initial investment is some other value, say K, then the amount function, which is just the amount at time t, is $A(t) = K \cdot a(t)$.

Insert Timeline

Ok, if you're paying attention to detail like I told you to, then you've noticed that I tried to throw one past you by inserting the word *constant* in the force of interest accumulation function above. Yes, we will kick it up a notch a little later in the manual and discuss non-constant force of interest, but for now all you need to know is that if you're given constant force of interest, δ, then that is equivalent to continuously compounding at the rate δ.

Now let's return to present value calculations. As was illustrated in the previous section when we introduced present (discounted) values, in order to discount a value back to time 0, we will *divide* by the accumulation function.

Important: When accumulating from time 0 to time t, *multiply* by the accumulation function $a(t)$. When discounting from time t back to time 0, *divide* by the accumulation function $a(t)$.

Insert Timeline

IMPORTANT: When using accumulation function notation, the time of the initial investment is $t = 0$. In order to generally uses accumulation functions to compute the accumulated value for an initial investment made at a time other than $t = 0$, we must first discount the initial investment to time $t = 0$ and then accumulate to the appropriate time. This is captured in the following timeline, whereby we are accumulating an amount Y, invested at time $t = k$, to time $t = n$.

Insert timeline

Y at time k to Z at time n

The same logic works if k is greater than n. In this case, we would say that Z is the *discounted* value of Y and the timeline would look like:
Insert timeline

For Examples 1-7 you are given:

$$a(1) = 1.2$$
$$a(2) = 1.5$$
$$a(3) = 2$$
$$a(4) = 3.0$$

Example 1: 100 is deposited at time $t = 0$. Determine the accumulated amount at time $t = 3$.

Solution 1: Insert timeline

Example 2: Determine the present value at time $t = 0$ of 75 at time $t = 2$.

Solution 2:

Example 3: The value at time $t = 1$ is 100. Determine the accumulated value at time $t = 4$.

Solution 3:

Example 4: Determine the discounted value at time $t = 2$ of a value of 400 at time $t = 3$.

Solution 4:

(Aside: In this manual, I make a slight distinction between the terms “present value” and “discounted value”. The term present value refers to the indifference value when an initial amount is being discounted to a focal point that is at or before time $t = 0$. In this context I’m thinking of the focal point as being the “present”. The term “discounted value” is more general and refers to the indifference value when an initial amount is being discounted to an earlier focal point, but the focal point is not necessarily at or before time $t = 0$. So a present value is a discounted value, but a discounted value may not be a present value. See the wordings of Examples 2 and 4. In Example 4 if we were computing the discounted value at time $t = 0$, then the result would be called the present value.)

Example 5: Determine the (total) present value at time $t = 0$ of the following two payments:
240 at time $t = 1$, plus 600 at time $t = 3$.

Solution 5:

Example 6: Determine the accumulated value at time $t = 4$ of the payments in Example 5.

Solution 6:

Example 7: Determine the value at time $t = 2$ of the payments in Example 5.

Solution 7: Just call it the value since one piece is an accumulation and the other is a discount.

Now that we know how to generally work with accumulation functions, let's consider the situation in the compounding case. Notice that the accumulation function in the compounding case is an exponential function; namely, $a(t) = (1+i)^t$ in the discrete compounding case and $a(t) = e^{\delta t}$ in the continuous compounding case. Since each of these functions is exponential, we can use properties of exponents to simplify ratios of accumulation functions. For example, an expression such as $\frac{a(7)}{a(5)}$ reduces to $\frac{(1+i)^7}{(1+i)^5} = (1+i)^2$ in the discrete case and $\frac{e^{7\delta}}{e^{5\delta}} = e^{2\delta}$ in the continuous case. Regardless, notice that the result is just $a(2)$.

Main point: When accumulating or discounting in the compounding case, the result only depends on the *time between* valuation dates and not necessarily on the *actual values of* the valuation dates.

See Examples 1 and 3 above. In both examples 100 is being accumulated for 3 periods. In Example 1 the accumulation is from time $t = 0$ to time $t = 3$ and the accumulated value is 200, whereas in Example 3 the accumulation is from time $t = 1$ to time $t = 4$ and the accumulated value is 250. In both cases the accumulation is for 3 periods, but the accumulated values are different!! This would not happen if we were in the compounding case. Let's do a few more examples.

Example 8: accumulating

Example 9: discounting

Example 10: continuous accumulation

Example 11: continuous discounting

Now let's do a couple of simple interest examples

Example 12: one payment accumulating

Example 13: one payment discounting

Example 14: two payment accumulation word ambiguously then example 14a and example 14b

Effective rates i_k

Develop this here

Unfortunate choice of words (simple interest case is complicated ; compound interest case is simple)
See now why use v notation in compounding situations only.

Equivalent rates:

Two rates are **equivalent** means that an investor would be indifferent between the two rates. For example, suppose Bank X advertises interest at an aeir of i on deposits, whereas Bank Y offers interest compounded continuously at rate δ. If you plan to deposit money into one of the accounts, would you be indifferent to the two choices if $i = \delta = 10\%$? If your answer is yes, then please call me at 867-5309 so that we can do business. You should not be indifferent between $i = 10\%$ and $\delta = 10\%$, since for every dollar invested, the accumulated value after one year will be more if interest is compounded continuously instead of being compounded just once during the year. The question is then, given i, what value of δ would make us indifferent to the banks' offers, or vice-versa, given δ, what value of i would make us indifferent. One way to answer this question is to recall that in this situation the annual discount factor is $v = \frac{1}{1+i} = e^{-\delta}$. Regarding the last equality, we can solve for either i or δ to get $\delta = \ln(1+i)$, or equivalently, $i = e^{\delta} - 1$. For example, if Bank X is offering interest on deposits at 10% compounded annually, then Bank Y should offer $\delta = \ln(1.10) \approx 9.531\%$ in order for investors to be indifferent between the two banks' offers. Similarly, if Bank Y is offering interest on deposits at 10% compounded continuously, then Bank X should offer an aeir of $i = e^{0.1} - 1 \approx 10.517\%$ in order for investors to be indifferent between the two banks' offers.

Generally, when determining equivalent compound rates, we can accumulate or discount an arbitrary amount (we'll use \$1) over an arbitrary period of time, set the resulting expressions equal to one another, and then solve. See the next several examples.

Examples

The situation is not so simple when simple interest is involved. This is because as we noted above, when using simple interest, accumulated and discounted values depend on the actual times at which the payments are valued, not just the amount of time between the valuation dates.

More equivalent rates problems (involving simple interest to show issues with simple interest) – one for one year and another for two years using same rate, both starting from time 0 and then another from time 2 to 5 years. Again see why simple is not so simple

Show simple interest over short amounts of time is better for investor than compounding.
(reason is linear vs exponential accumulation functions)