Module 1: Interest and Discount

Section 1: Simple and Compound Interest

The whole premise behind Interest Theory is that there is a **time value of money**. This means that a given amount of money has a different value depending on when the amount is under consideration. For example, in order to buy the same amount of beer today that I bought in 1990 with $20 would require much more than $20. The time value of money is accounted for through interest. In the problems in this manual you'll be given an amount or amounts at one or more points in time and asked to value the amount(s) at another point in time, called the **valuation date (point)**. If the valuation date is later in time than the time the initial value is given, then we are **accumulating** the initial amount. If the valuation date is earlier in time than the initial value is given, then we are **discounting** the initial amount. The value that's being computed is an **indifference** value; it's the value that, if two parties were trading the amounts at the different times, both parties would be indifferent to the trade. The indifference value is usually not given directly, but is to be computed. The calculated amount will generally depend on the rate at which interest is being calculated and the times at which the amounts are given.

Many of you will be tempted to skip this section since it is material that you basically learned in middle school. Please don't. I will be introducing terminology, notation, and concepts that you need to master before you move on to later sections of the manual. Although you may be thinking that I'm including way too much detail on such a simple topic as, say, simple interest and compound interest, there is a reason to my madness. Trust me, continue reading and working the examples, and then I think you'll understand later.

Let's start with an example. Before looking at the solution, take a minute to determine your answer.

Example 1: 100 is invested in an account for 2 years at an interest rate of 20%. Determine the amount in the account (indifference value) after the 2 year period.

Solution: Wow! Where do we begin? After you've completed this section of the manual, you'll look back on this problem and see how poorly worded it is. First of all, there's no period given with the interest rate. Is the 20% rate an annual rate, or monthly rate, or some other periodic rate? Even if it is assumed (btw, not a good idea – you know the whole ass, u, me thing …), but even if it is assumed the rate is an annual rate, is it a simple interest rate or a compound interest rate?

This example illustrates a problem that actuaries run into in their daily work, interpreting ambiguous statements. We, as actuaries, pay attention to details, but sometimes plan documents or laws and such are not worded very well. Later in your career it will be your job to interpret the laws, plan documents, etc. However, at this point in your career, your job is to pass actuarial exams. The material in actuarial exams is developed in such a way as to reduce ambiguities. The first part of this manual (Interest Theory) is a prime example of detailing concepts to avoid ambiguity.

Let's return to the above example. Actually, let's work the example in two ways; using simple interest (Example 1a) and then using compound interest (Example 1b). The first thing to do is word the problem properly.

Example 1a: 100 is invested in an account for 2 years at a simple interest rate of 20%. Determine the amount in the account after the 2 year period.

(Aside: Notice that the inclusion of the word simple is the only difference in the wording between Example 1 and Example 1a. This is a key word in the problem, though, since its omission would make the problem ambiguous. As an actuarial exam student, you must learn to pick up on the key words of each problem in order to be successful.)

Solution 1a: Unless told otherwise, simple interest rates are always annual rates. Recall from algebra class the formula for calculating interest: $I = Prt$. We will denote simple interest rates by the letter *i*, rather than *r*. Therefore our formula becomes $I = Pit$.

(Aside: Generally lower case letters will denote rates and capital letters will denote amounts.)

Since the interest rate is an annual rate, we separate the overall time period (2 years in this case) into annual subperiods and use *t* = 1 in the calculation of the amount of interest earned in each subperiod. Now recall that simple interest means that interest is earned on principal only. This means that the principal remains constant in each of the annual subperiods. In this example we are given *i* = 20% simple and *P* = 100. Consequently, the amounts of interest earned in the first and second years, denoted by I_1 and I_2, respectively, are $I_1 = I_2 = 100(0.2)(1) = 20$, and the total amount of interest earned over the entire 2 year period is $I = I_1 + I_2 = Pi + Pi = 20 + 20 = 40$. The total amount, *A*, in the account at the end of two years is $A = P + I_1 + I_2 = 100 + 20 + 20 = 140$. This can all be captured in a timeline as follows:

Insert timeline

Finally, notice that since the principal remains constant in each subperiod, the amount of interest earned in each subperiod is Pi (constant), and the total amount of interest earned during the entire timeframe is $I = \sum I_k = \sum Pi = Pit$. Therefore the total amount in the account at the end of the entire timeframe is $A = P + I = P + Pit = P(1 + it)$.

(Aside: Although our example was for a short 2 year period, the above derivation of the amount formula for simple interest is valid for any time period. Make sure that time is calculated in years since the simple interest rate is an annual rate.)

Now let's rework (and reword) the problem using compound interest. I'll introduce some new terminology and notation in this example. Pay attention to the details.

Example 1b: 100 is invested in an account for 2 years at an interest rate of 20%, compounded annually. Determine the amount in the account after the 2 year period.

Solution 1b: Since the interest is compounded annually, we again separate the overall time period (2 years) into annual subperiods. We call *i* an **annual effective interest rate** (abbreviated aeir). Recall that interest is compounded means interest is being earned on interest. Therefore in our formula for interest, $I = Prt$, the principal amount is changing each period (year). So we use subscripts on principal. We get $I_1 = P_0 i = 100(0.2) = 20$, as the simple interest case. However, for I_2 we have $I_2 = P_1 i$ where $P_1 = P_0 + I_1 = 120$ is the principal amount used for the second year. Therefore

$I_2 = 120(0.2) = 24$ the total amount in the account at the end of two years is $A = P_0 + I_1 + I_2 = 100 + 20 + 24 = 144$. Again, this can all be captured in a timeline as follows:

Insert timeline

Using subscripts for the principal at different times, we have the formula for the amount at the end of the two year period: $A = P + I = P_0 + I_1 + I_2 = P_0 + P_0 i + P_1 i$. Noting that $P_1 = P_0 + I_1 = P_0 + P_0 i = P_0(1+i)$, the above amount formula simplifies to $A = P_1 + P_1 i = P_1(1+i) = P_0(1+i)^2$. This formula can be easily generalized to $A = P(1+i)^t$, where P is the original principal amount P_0, *i* is the aeir and *t* is the length of time of the investment in years. As in the simple interest situation above, the formula is valid for any value of *t*.

Now consider a similar problem, except that we change the compounding frequency. New terminology and actuarial notation is introduced in this example, which needs to be mastered.

Example 2: 100 is invested in an account for 2 years at an interest rate of 20%, compounded semiannually. Determine the amount in the account after the 2 year period.

Solution 2: The only difference is that this time the interest is compounded semiannually. The quoted rate of 20% is called a **nominal interest rate**. Since the nominal rate is compounded twice per year, the actuarial notation for this nominal rate is $i^{(2)} = 20\%$. This is read "*i* upper 2", and it is important to use the parenthesis around the 2 in order to distinguish this value from the different value of "*i* squared". Our goal is to compute an accumulated value after two years, and in such a situation, the first step is to compute the **periodic effective interest rate** (eir) . This is done by dividing the nominal rate by the number of compounding periods per year, which is the number in parenthesis when writing the nominal rate using actuarial notation. In this case we have $i = \frac{i^{(2)}}{2} = \frac{0.2}{2} = 0.1$, the semiannual eir (seir). This is the actual numeric value that will be used in our calculations. Now we proceed as we did in Example 1b, except that we use semiannual subperiods over the entire two year period. As there are 4 semiannual periods over the two years, we have $A = P + I_1 + I_2 + I_3 + I_4$. Similar to before, $I_1 = Pi$, $I_2 = P_1 i$, $I_3 = P_2 i$, and $I_4 = P_3 i$, and so we get $I_1 = 100(0.1) = 10$, $I_2 = 110(0.1) = 11$, $I_3 = 121(0.1) = 12.1$, and $I_4 = 133.1(0.1) = 13.31$. Therefore $A = 100 + 10 + 11 + 12.1 + 13.31 = 146.41$. Again, this can all be captured in a timeline as follows:

Insert timeline

Just as we did in Example 1b, we can derive another formula for the amount after two years as follows: $A = P_3(1+i) = P_2(1+i)^2 = P_1(1+i)^3 = P_0(1+i)^4$. As before, this formula can be easily generalized to $A = P(1+i)^t$, where P is the original principal amount P_0, *i* is the seir and *t* is the length of time of the investment in semiannual periods.

Finally, we generalize to any compounding period as follows: If P is invested at a periodic eir of *i* then the accumulated amount at the end of *t* periods is $A = P(1+i)^t$.

Insert timeline

Aside: You may recall from algebra the amount formula $A = P(1 + \frac{r}{n})^{nt}$ for compound interest. In this formula, r is what we now call the nominal interest rate compounded n times per year, denoted by $i^{(n)}$, and t is always in *years*. Notice this is the same formula as our $A = P(1+i)^t$, where in our formula $i = \frac{i^{(n)}}{n}$ is the periodic eir and t is the number of *periods* of the investment. Be careful to "match the periods" on the periodic eir i and the length of the investment t. For example, if i is an meir (monthly eir), then the exponent t should equal the number of months of the investment.

Several examples follow. Try completing the examples yourself before looking at the solutions that follow.

Example 3: An interest rate is given as 18%, compounded monthly.

(a) The 18% rate is called a _____ ______ ______. The actuarial notation for this rate is _____ and is read _____ _____ _____.

(b) In calculating an accumulated amount using this rate, the first step is to ______ this rate by _____. The resulting rate is called the _____ ______ _____ _____ and in this manual we will abbreviate that by _____.

(c) In calculating an accumulated amount using this rate, the exponent should be time measured in _____.

Solution 3: The 18% rate is called a nominal interest rate. The actuarial notation for this rate is $i^{(12)}$ and is read i upper 12. In calculating an accumulated amount using this rate, the first step is to divide this rate by 12. The resulting rate, $i = \frac{i^{(12)}}{12} = \frac{0.18}{12} = 0.015$, is called the monthly effective interest rate and in this manual we will abbreviate that by meir. The exponent in the amount calculation should be time measured in months.

The next several examples (Examples 4 through 10) illustrate a certain phenomenon that takes place with compound interest.

Example 4: 1000 is invested in an account for 5 years at an interest rate of 12%, compounded annually. Determine the amount in the account after the 5 year period.

Solution 4: We have $i = 0.12 = \text{aeir}$, and $t = 5$ years. Therefore $A = 1000(1.12)^5 = 1762.34$.

Example 5: 1000 is invested in an account for 5 years at an interest rate of 12%, compounded semiannually. Determine the amount in the account after the 5 year period.

Solution 5: We have $i = 0.06 = \text{seir}$, and $t = 12$ semiannual periods. Therefore $A = 1000(1.06)^{10} = 1790.85$.

Example 6: 1000 is invested in an account for 5 years at an interest rate of 12%, compounded quarterly. Determine the amount in the account after the 5 year period.

Solution 6: We have $i = 0.03 = \text{qeir}$, and $t = 20$ quarters. Therefore $A = 1000(1.03)^{20} = 1806.11$.

Example 7: 1000 is invested in an account for 5 years at an interest rate of 12%, compounded monthly. Determine the amount in the account after the 5 year period.

Solution 7: We have $i = 0.01 = \text{meir}$, and $t = 60$ months. Therefore $A = 1000(1.01)^{60} = 1816.70$.

Example 8: 1000 is invested in an account for 5 years at an interest rate of 12%, compounded daily. Determine the amount in the account after the 5 year period. Assume 365 days per year.

Solution 8: We have $i = \frac{0.12}{365} =$ deir, and $t = 1{,}825$ days. Therefore $A = 1000(1 + \frac{0.12}{365})^{1825} = 1821.94$.

Example 9: 1000 is invested in an account for 5 years at an interest rate of 12%, compounded 1000 times per year. Determine the amount in the account after the 5 year period.

Solution 9: We have $i = \frac{0.12}{1000} = \text{periodic eir}$, and $t = 5000$ periods. Therefore $A = 1000(1.00012)^{5000} = 1822.05$.

Example 10: 1000 is invested in an account for 5 years at an interest rate of 12%, compounded 10,000 times per year. Determine the amount in the account after the 5 year period.

Solution 10: We have $i = \frac{0.12}{10000} = \text{periodic eir}$, and $t = 50{,}000$ periods. Therefore $A = 1000(1.000012)^{50000} = 1822.11$.

Notice that the amounts after 5 years in the examples are increasing, as expected since compounding (interest being earned on interest) takes place more and more frequently. However, the increase in the amounts is not without bound. There seems to be a limiting value, and in fact, using limit facts from calculus, we can see exactly what that limiting value is. Let n denote the number of compounding periods per year. Recalling from calculus that $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$, and noting there are $5n$ periods in 5 years, and we have $A = 1000(1 + \frac{0.12}{n})^{5n} = 1000[(1 + \frac{.12}{n})^n]^5 \to A = 1000e^{0.12(5)} = 1822.12$ as $n \to \infty$. This is nothing more than the familiar "continuously compounded interest" formula $A = Pe^{rt}$ (recall that t is measured in years). There is another term and notation that we actuaries use in this situation. Although we do refer to continuously compounded interest, in the context of interest theory we use the lower case Greek delta to represent this rate, and we call it the **force of interest**. So when you see $\delta = 0.12$, then that is equivalent to the statement that interest is compounded continuously at the nominal annual rate of 12%. We could even write this as $\delta = i^{(\infty)}$.

SUMMARY:

i-simple interest rate, t-time in years	$A = P(1 + it)$
i-periodic eir, t-number of periods	$A = P(1 + i)^t$
δ-force of interest, t-time in years	$A = Pe^{\delta t}$

Since we've solved for A in the above formulas, we are thinking of the principal as being given and we're calculating an accumulated value. We can easily solve the above formulas for P, and then we would be thinking of the accumulated value as being given and we're calculating the principal. In this context we call the principal the **present (or discounted) value**. We get:

(Simple) $P = \frac{A}{1+it}$

(Compounding) $P = A(1 + i)^{-t}$ (or $P = Ae^{-\delta t}$ for continuous compounding)

When calculating present values in the context of compounding, actuaries use what is called **v (vee) notation**, where v is called the **periodic discount factor** (v is the annual discount factor in the case of continuous compounding since time is in years in this case). The idea is that v equals the discounted value of \$1 for one period. It follows that v^t is the discounted value of \$1 for t periods, and we get the simple equation $P = Av^t$ for the present value when compounding. Note that the periodic discount factor is $v = (1+i)^{-1} = \frac{1}{1+i}$ where i is the periodic eir, and in the case of continuous compounding, $v = e^{-\delta}$.

Insert timeline

(Aside: We do not use v notation in the simple context, only in the compounding context. Note that the periodic discount factor is the reciprocal of $1 + i$, which is called the **periodic accumulation factor**.)

Examples (where appropriate, use v notation)

Exercises:
Like examples
Give timeline and ask for question