Section 1 – Interest and Discount

Accumulation Function – denoted by $a(t)$ and gives the accumulated value at time t of 1 invested at time 0.

Amount Function – denoted by $A(t)$ and gives the accumulated value at time t of k invested at time 0.

Present Value Function – denoted by $PV_0 = PV_0(t)$ gives the present value at time 0 of h at time t. This can be thought of as the amount that must be invested at time 0 that will accumulate to h at time t.

Basic Formulas:

$$a(0) = 1$$
$$A(0) = k$$
$$A(t) = k \cdot a(t)$$
$$PV_0 = \frac{h}{a(t)} = h \cdot [a(t)]^{-1}$$

Our ultimate goal is to calculate accumulated and present values under different assumptions.

Interest Amounts, Effective Rates, and Nominal Rates of Interest

The amount of interest earned during the period from time *t*-1 to time *t* is

$$I_t = A(t) - A(t-1)$$

The effective interest rate (eir) during this period is

$$i_t = \frac{I_t}{A(t-1)} = \frac{A(t) - A(t-1)}{A(t-1)} = \frac{a(t) - a(t-1)}{a(t-1)}$$

Common periods used are monthly, quarterly, semiannually, and annually. A special situation arises in practice in which a nominal (quoted) rate of interest is given along with how frequently interest is compounded.

Example: If interest is quoted as 6% compounded monthly, then we would write

$$i^{(12)} = .06$$

and 6% is called the nominal rate of interest. We then get the monthly effective interest rate (meir) as follows

$$i = \frac{i^{(12)}}{12} = \frac{.06}{12} = .005 = .5\% \text{ (meir)}$$

Examples of Formulas and Notation

$$i = \frac{i^{(12)}}{12} = meir$$

$$i = \frac{i^{(4)}}{4} = qeir$$

$$i = \frac{i^{(2)}}{2} = seir$$

Discount Amounts, Effective Rates, and Nominal Rates of Discount

The amount of discount during the period from time *t*-1 to time *t* is the same as the amount of interest during the period; namely,

$$I_t = A(t) - A(t-1)$$

The difference between discount and interest is that with interest, the interest amount is paid at the end of the period and is based on the amount function at the beginning of the period, whereas with discount, the discount amount (which is the same as the interest amount) is paid at the beginning of the period and is based on the amount function at the end of the period.

Formula for the periodic effective discount rate (periodic edr):

$$d_t = \frac{I_t}{A(t)} = \frac{A(t) - A(t-1)}{A(t)} = \frac{a(t) - a(t-1)}{a(t)}$$

Analogous to compound interest, there is compound discount. Usually the nominal rate of discount is given along with compounding frequency. Example: If the discount rate is 6% compounded monthly, then we write

$$d^{(12)} = .06$$

and 6% is called the nominal rate of discount. We then get the monthly effective discount rate (medr) as follows

$$d = \frac{d^{(12)}}{12} = \frac{.06}{12} = .005 = .5\% \text{ (medr)}$$

Examples of Formulas and Notation

$$d = \frac{d^{(12)}}{12} = medr$$

$$d = \frac{d^{(4)}}{4} = qedr$$

$$d = \frac{d^{(2)}}{2} = sedr$$

Force of Interest – (foi) is denoted at time *t* by δ_t and is defined by

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{a'(t)}{a(t)} = \frac{d}{dt}[\ln(a(t))]$$

Calculating Accumulation and Present Value Functions

Interest Type	Accumulated Value	Present Value
Simple Interest *i* = simple interest rate	$a(t) = 1 + i \cdot t$	$PV_0 = \frac{1}{1+i\cdot t} = (1+i\cdot t)^{-1}$
Simple Discount *d* = simple discount rate	$a(t) = \frac{1}{1-d\cdot t} = (1-d\cdot t)^{-1}$	$PV_0 = 1 - d\cdot t$
Compound Interest *i* = eir	$a(t) = (1+i)^t$	$PV_0 = (1+i)^{-t} = v^t$
Compound Discount *d* = edr	$a(t) = (1-d)^{-t}$	$PV_0 = (1-d)^t = v^t$
Force of Interest δ_t = foi	$a(t) = \exp\left(\int_0^t \delta_s ds\right)$	$PV_0 = \exp\left(-\int_0^t \delta_s ds\right)$

Notes:

1. $v = (1+i)^{-1} = 1-d$ is called the interest discount factor
2. If $\delta_t = \delta$ (a constant) then $a(t) = e^{\delta\cdot t}$ (continuous compounding).
3. If $\delta_t = c\cdot\frac{f'(t)}{f(t)}$, then $a(t) = \left(\frac{f(t)}{f(0)}\right)^c$.
4. Unless told otherwise, assume interest is compounded.
5. In order to determine the equivalent rate of interest or discount to a given rate, either accumulate or discount an amount, say 1, over some fixed period of time using both rates, and then set the two expressions equal and solve.

Relating *i, d,* ν, and δ.

Master the following relationships among *i, d,* ν, and δ.

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\begin{aligned}
d &= i \cdot \nu \\
\nu &= 1 - d = \frac{1}{1+i} = e^{-\delta} \\
d &= 1 - \nu = \frac{i}{1+i} = 1 - e^{-\delta} \\
i &= \frac{d}{1-d} = e^{\delta} - 1 = \nu^{-1} - 1 \\
\delta &= \ln(1+i) = -\ln(1-d) = -\ln(\nu)
\end{aligned}
$$

Method of Equated Time – approximates the time in which a single payment of $\sum_{i=1}^{n} s_i$ is equivalent to payments of s_i at time t_i for $i = 1, 2, \cdots, n$.

Formula for Calculating Method of Equated Time

$$\bar{t} = \frac{\sum_{i=1}^{n} s_i \cdot t_i}{\sum_{i=1}^{n} s_i}$$

Note that this is just equal to the weighted average of the payment times, where the weight at each payment time is the ratio of the amount of the payment at that time to the total of all payments.

The following concepts of **equations of value, internal rates of return, and net present values** will be illustrated by working examples.