1. (a) $\sigma = (1\,2\,5)(2\,4)^{-1}(1\,3\,5)(3\,4\,5)^{-1} = (1\,2\,5)(2\,4)(1\,3\,5)(3\,5\,4) = (1\,3\,2\,4)$.
(b) The decomposition of σ into disjoint cycles has the form $(a\ b)(c\ d\ e)$.
(c) There is no such σ, because $(1\ 2)(3\ 4\ 5)$ is odd and σ^2 is even for all $\sigma \in S_5$.

2. For all $a, b \in G$, $a^2 = b^2 = (ab)^2 = e$, hence $a^{-1} = a$, $b^{-1} = b$, and $(ab)^{-1} = ab$. But $(ab)^{-1} = b^{-1}a^{-1}$, so $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$.

3. (a) By the Structure Theorem for finite(ly generated) abelian groups, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$.
(b) (i) G is nonabelian, hence (I) G is not cyclic, so no element has order 6, and (II) G has a nonidentity element a of order $\neq 2$ by Problem 2. The order of a is a divisor of $|G| = 6$, so a must have order 3. (ii) Because $(G : N) = 6/3 = 2$. (iii) N has two cosets: $N = \{e, a, a^2\} = \{a^0b^0, a^1b^0, a^2b^0\}$ itself and $Nb = \{b, ab, a^2b\} = \{a^0b^1, a^1b^1, a^2b^1\}$. (iv) $bab^{-1} \in bNb^{-1} \subseteq N = \{e, a, a^2\}$, hence $bab^{-1} = e$ or $bab^{-1} = a$ or $bab^{-1} = a^2$. The first equality is impossible, because it implies that $a = b^{-1}eb = e$. The second equality is impossible, because it implies (together with statement (iii)) that G is abelian. (v) Nb generates the 2-element group G/N, hence has order 2. Therefore the order r of b is a multiple of 2. Since $r \neq 6$, we must have $r = 2$. (vi) The properties in (iii)–(v) uniquely determine multiplication in G. For example, $(ab)(a^2b) = a(ba)ab = a(a^2b)ab = a^3bab = bab = a^2bb = a^2$. By completing the multiplication tables for G and for S_3 one can see that the function $G \to S_3$, $a^ib^j \mapsto (1\ 2\ 3)^i(1\ 2)^j$ $(0 \leq i \leq 2,\ 0 \leq j \leq 1)$ is an isomorphism.

4. Let $a \in G$, $a \neq e$. Since G has no nontrivial proper subgroups, $\langle a \rangle = G$. The order r of a cannot be infinite, because then $\langle a^2 \rangle$ would be a nontrivial proper subgroup of $\langle a \rangle = G$. Hence $r = |G|$ is an integer > 1. If r is not prime, say $r = dm$ with $d, m > 1$ $(d, m \in \mathbb{Z}^+)$, then again $\langle a^d \rangle$ is a nontrivial proper subgroup of $\langle a \rangle = G$. Hence $r = p$ is prime and $G \cong \mathbb{Z}_p$.

5. (a) By an earlier HW, $\langle \sigma^2 \rangle$ is a normal subgroup of D_4. If $\langle \mu \rangle$ is another 2-element subgroup of D_4, then $\sigma\mu\sigma^{-1} \notin \langle \mu \rangle$, therefore $\langle \mu \rangle$ is not normal.
(b) Let $H = \langle \sigma \rangle$. Then $D_4/H = \{H, \sigma H, \mu_1 H, \mu_2 H\}$ where $\mu_1^2 = \mu_2^2 = \iota$. Thus $(\mu_i H)^2 = \mu_i^2 H = \iota H = H$ for $i = 1, 2$, and $(\sigma H)^2 = \sigma^2 H = H$ because $\sigma^2 \in H$.
(c) Assume there exists an onto homomophism $\varphi\colon D_4 \to \mathbb{Z}_4$. Then $\mathbb{Z}_4 = \varphi[D_4] \cong D_4/N$ for some normal subgroup N of D_4 of order 2. By part (a), $N = \langle \sigma \rangle$, hence by part (b), $D_4/N \ncong \mathbb{Z}_4$, a contradiction.

6. (a) True. If $\sigma^r = \iota$ with r odd, then σ, σ^r, ι have the same parity: even.
(b) False. If $\varphi\colon \mathbb{Z}_{10} \to \mathbb{Z}_{21}$ is a homomorphism, then $\varphi[\mathbb{Z}_{10}] \leq \mathbb{Z}_{21}$ and $\varphi[\mathbb{Z}_{10}] \cong \mathbb{Z}_{10}/N$ for a (normal) subgroup $N \leq \mathbb{Z}_{10}$. Thus $r := |\varphi[\mathbb{Z}_{10}]|$ is a common divisor of $|\mathbb{Z}_{21}| = 21$ and $|\mathbb{Z}_{10}| = 10$, hence $r = 1$.
(c) True. Since 29 is prime, all groups of order 29 are isomorphic to $\mathbb{Z}_{29}$.
(d) True. Let $a \in H$. Since a has finite order r and H is closed under multiplication, $e = a^r = aaa \cdots a \in H$ and $a^{-1} = a^{r-1} = aa \cdots a \in H$.