Some questions on the exam may ask you to state definitions or theorems.

Practice Problems

1. For each property below find all $\sigma \in S_5$ that satisfy the given property:
(a) $(2\ 4)(1\ 2\ 5)^{-1}\sigma\,(3\ 4\ 5) = (1\ 3\ 5)$;
(b) σ has order 6;
(c) $\sigma^2 = (1\ 2)(3\ 4\ 5)$.

2. Show that if every nonidentity element of a group G has order 2, then G is abelian. [*Hint:* Use the assumption for $a, b, ab \in G$.]

3. Let G be a group of order 6.
(a) Prove that if G is abelian, then $G \cong \mathbb{Z}_6$;
(b) Now assume that G is nonabelian, and show that
(i) G has an element a of order 3; [*Hint:* Use Problem 2.]
(ii) $N = \langle a \rangle$ is a normal subgroup of G;
Let $b \in G$ be such that $b \notin N$.
(iii) every element of G is of the form $a^i b^j$ $(0 \le i \le 2,\ 0 \le j \le 1)$, and this form is unique; [*Hint:* Think of the cosets of N.]
(iv) $bab^{-1} = a^2$, that is, $ba = a^2 b$;
(v) b has order 2; [*Hint:* First find the order of bN in G/N.]
(vi) $G \cong S_3$.

4. Prove that if a group G of order > 1 has no nontrivial proper subgroups, then $G \cong \mathbb{Z}_p$ for some prime p. [*Hint:* Argue first that G is cyclic.]

5. Let σ be one of the elements of order 4 in D_4.
(a) Show that N is a normal subgroup of D_4 of order 2 if and only if $N = \langle \sigma^2 \rangle$.
(b) Show that every nonidentity element of $D_4/\langle \sigma^2 \rangle$ has order 2.
(c) Use the Homomorphism Theorem to prove that D_4 has no homomorphism onto $\mathbb{Z}_4$.

6. TRUE or FALSE? Justify your answer.
(a) *If $\sigma \in S_n$ has odd order, then σ is an even permutation.*
(b) *There exists a nontrivial homomorphism $\mathbb{Z}_{10} \to \mathbb{Z}_{21}$.*
(c) *Any two groups of order 29 are isomorphic.*
(d) *If G is a finite group and H is a nonempty subset of G such that H is closed under multiplication, then H is a subgroup of G.*