1. Since $A \in P(A)$, the inclusion $P(A) \subseteq P(B)$ implies that $A \in P(B)$, hence $A \subseteq B$.

2. (a) $(a,b) \in \sigma$ if and only $b^2 = a^2 + 1$ and $b \leq 0$, which holds if and only if $b = -\sqrt{a^2+1}$. Therefore σ determines a function $\mathbb{R} \to \mathbb{R}$. It is not one-to-one, since $\sigma(1) = \sigma(-1)$, and is not onto, since $0 \notin \sigma[\mathbb{R}]$.

(b) $(a,b) \in \tau$ if and only $b^2 = a^2 - 1$ and $b \geq 0$, therefore τ contains no pair (a,b) with $a = 0$. Hence τ is not a function $\mathbb{R} \to \mathbb{R}$.

3. (a) Assume that f is one-to-one and $f \circ g = f \circ h$. For arbitrary $c \in C$, $(f \circ g)(c) = (f \circ h)(c)$, so by the definition of composition, $f(g(c)) = f(h(c))$. Since f is one-to-one, we get that $g(c) = h(c)$ for all $c \in C$. This proves that $g = h$.

(b) If f is not one-to-one, then there exist distinct elements $a, a' \in A$ such that $f(a) = f(a')$. Define $g, h\colon \mathbb{Z}_2 \to A$ such that $g(0) = g(1) = a$ and $h(0) = a$, $h(1) = a'$. Clearly, $g \neq h$. However, $f \circ g = f \circ h$, because $f(g(0)) = f(a) = f(h(0))$ and $f(g(1)) = f(a) = f(a') = f(h(1))$.

4. Since f is a bijection, it has an inverse function f^{-1}. Now using the equality $f \circ g = \iota_B$ and basic properties of composition we get that $g = \iota_A \circ g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g) = f^{-1} \circ \iota_B = f^{-1}$.

5. (a) (i) $\sim$ is reflexive: For arbitrary pair $(a_1, a_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ we have $a_1 a_2 = a_2 a_1$, proving $(a_1, a_2) \sim (a_1, a_2)$. (ii) $\sim$ is symmetric: If $(a_1, a_2) \sim (b_1, b_2)$, that is, $a_1 b_2 = a_2 b_1$, then also $b_1 a_2 = b_2 a_1$, hence $(b_1, b_2) \sim (a_1, a_2)$. (iii) $\sim$ is transitive: If $(a_1, a_2) \sim (b_1, b_2)$ and $(b_1, b_2) \sim (c_1, c_2)$, that is, $a_1 b_2 = a_2 b_1$ and $b_1 c_2 = b_2 c_1$, then $(a_1 b_2)(b_1 c_2) = (a_2 b_1)(b_2 c_1)$, so dividing by $b_1 b_2 > 0$ yields $a_1 c_2 = a_2 c_1$, whence $(a_1, a_2) \sim (c_1, c_2)$.

(b) The partition is the set of all equivalence classes $\overline{(a_1, a_2)} = \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid a_1 y = a_2 x\} = \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid y = (a_2/a_1)x\}$, i.e., the set of all rays in the first quadrant starting at (but not including) the origin.

(c) $f\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $f(x_1, x_2) = x_2/x_1$.

6. (a) $*$ is commutative if and only if $r = s$. Indeed, $x * y = y * x$ for all $x, y \in \mathbb{R}$ if and only if $rx + sy = ry + sx$ for all $x, y \in \mathbb{R}$, which is false if $r \neq s$ (e.g. let $x = 1$, $y = 0$), and is true if $r = s$.

(b) $*$ is associative if and only if $r, s \in \{0, 1\}$. Indeed, $(x*y)*z = x*(y*z)$ for all $x, y, z \in \mathbb{R}$ if and only if $r^2x + rsy + sz = rx + sry + s^2z$ for all $x, y, z \in \mathbb{R}$, which is false unless $r^2 = r$ and $s^2 = s$ (substitute $x = 1, y = z = 0$ and $x = y = 0, z = 1$), and is true if $r^2 = r$ and $s^2 = s$.

7. φ is one-to-one, since it is strictly increasing, and φ is onto, because for all $y \in \mathbb{R}_{2\pi}$ we have $y/2 \in \mathbb{R}_\pi$ and $\varphi(y/2) = y$. Now let $x, y \in \mathbb{R}_\pi$. If $x + y < \pi$, then $2x + 2y < 2\pi$, so $\varphi(x +_\pi y) = 2(x + y) = 2x + 2y = \varphi(x) +_{2\pi} \varphi(y)$, while if $x + y \geq \pi$, then $2x + 2y \geq 2\pi$, so $\varphi(x +_\pi y) = 2(x + y - \pi) = 2x + 2y - 2\pi = \varphi(x) +_{2\pi} \varphi(y)$. Thus φ is an isomorphism.

8. (a) False. For example, $f \circ g = \iota_{\mathbb{Z}}$ holds for $f\colon \mathbb{Z} \to \mathbb{Z}$, $n \mapsto \lfloor n/2 \rfloor$, and $g\colon \mathbb{Z} \to \mathbb{Z}$, $n \mapsto 2n$, but g is not bijective, so it has no inverse function.

(b) True. $f \circ g = \iota_A$ is one-to-one, hence so is g. (See Thm 2 on Handout 1.)

(c) False. For example, the function $\mathbb{Z} \to \mathbb{Z}$, $n \to n+1$, is not an isomorphism of $(\mathbb{Z}, +)$ with itself, but $(\mathbb{Z}, +) \cong (\mathbb{Z}, +)$ ($\iota_{\mathbb{Z}}$ is an isomorphism).