Some questions on the exam may ask you to state definitions or theorems.

Practice Problems

1. For a set A let $\mathcal{P}(A)$ be the set of all subsets of A (called the **power set** of A). Show that if A and B are sets such that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

2. For each subset of $\mathbb{R} \times \mathbb{R}$ below decide whether or not it determines a function $\mathbb{R} \to \mathbb{R}$. If it does, is the function one-to-one? Is it onto?
(a) $\sigma = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid b^2 - a^2 = 1,\ b \leq 0\}$,
(b) $\tau = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid b^2 - a^2 = -1,\ b \geq 0\}$.

3. Let $f\colon A \to B$ be an arbitrary function. Prove that
(a) if f is one-to-one and $g, h\colon C \to A$ are such that $f \circ g = f \circ h$, then $g = h$.
(b) if f is not one-to-one, then there exist functions $g, h\colon \mathbb{Z}_2 \to A$ such that $f \circ g = f \circ h$ and $g \neq h$.

4. Show that if $f\colon A \to B$ is a bijection and $g\colon B \to A$ is a function such that $f \circ g = \iota_B$, then $g = f^{-1}$.

5. Let $\sim$ be the relation on the set $\mathbb{R}^+ \times \mathbb{R}^+$ defined by

$$(a_1, a_2) \sim (b_1, b_2) \quad \text{if and only if} \quad a_1 b_2 = a_2 b_1.$$

(a) Prove that $\sim$ is an equivalence relation.
(b) Visualizing the elements of $\mathbb{R}^+ \times \mathbb{R}^+$ as points in the first quadrant, give a geometric description of the partition corresponding to $\sim$.
(c) Find a function with domain $\mathbb{R}^+ \times \mathbb{R}^+$ whose kernel is $\sim$.

6. Let $r, s \in \mathbb{R}$, and define an operation $*$ on $\mathbb{R}$ by $x * y = rx + sy$ for all $x, y \in \mathbb{R}$. For which $r, s \in \mathbb{R}$ is the operation $*$ (a) commutative, (b) associative?

7. Show that the function $\varphi\colon \mathbb{R}_\pi \to \mathbb{R}_{2\pi}$ defined by $\varphi(x) = 2x$ for all $x \in \mathbb{R}$ is an isomorphism of $(\mathbb{R}_\pi, +_\pi)$ with $(\mathbb{R}_{2\pi}, +_{2\pi})$.

8. TRUE or FALSE? Justify your answer.
(a) *If $f, g\colon A \to A$ are functions such that $f \circ g = \iota_A$, then $f = g^{-1}$.*
(b) *If $f, g\colon A \to A$ are functions such that $f \circ g = \iota_A$, then g is one-to-one.*
(c) $(S, *) \not\cong (S', *')$ *if there exists a function $\varphi\colon S \to S'$ that is not an isomorphism between $(S, *)$ and $(S', *')$.*