All questions carry equal weight. State answers clearly and carefully, and justify all assertions with proofs or counterexamples. You may not use any books or notes.

(1) Suppose G is a group, and N a normal subgroup, not equal to all of G. Suppose that there are no subgroups H of G containing N and not equal to N or G. Show that N has finite index in G, and this index is a prime number.

(2) Let G be a group of order 12.

(a) Show that the number of subgroups of order 3 is either 1 or 4.

(b) Suppose that if there are four subgroups of order 3, and write them $P_1, \ldots, P_4$. Show that if $i \neq j$, then $P_i \cap P_j = \{e\}$.

(c) Continuing from (b), show that $G \setminus (P_1 \cup \ldots P_4) \cup \{e\}$ is a subgroup of G having order 4.

(d) Conclude that G either has a normal subgroup of order 3, or of order 4.

(3) Show that no commutative ring has its underlying additive group isomorphic to $\mathbb{Q}/\mathbb{Z}$.

(4) Show that if R is an integral domain, and $G \subset (R^\times, \cdot)$ which is a multiplicative group of finite order n, then G must be cyclic. Hints: consider the least common multiple of the orders of elements of G. Also consider the roots of the polynomials $x^d - 1$ in $R[x]$.