

All questions carry equal weight. State answers clearly and carefully, and justify all assertions with proofs or counterexamples. You may not use any books or notes.

(1) Describe 8 non-isomorphic groups of order 36.

Solution: consider the 8 groups $C_{36}, C_2 \times C_2 \times C_9, C_4 \times C_3 \times C_3, C_2 \times C_2 \times C_3 \times C_3, C_6 \times S_3, C_3 \times A_4, C_2 \times D_9, S_3 \times S_3$, where we abbreviate $\mathbb{Z}/n\mathbb{Z}$ by C_n . It is clear that these all have 36 elements, and we claim that they are pairwise non-isomorphic. The first four are abelian, and the second four aren't, so we may consider the first four and second four groups separately. We use repeatedly the fact that orders of $(a, b) \in G_1 \times G_2$ are given by $\text{lcm}(o(a), o(b))$.

None of the first four groups are isomorphic because the first is the only one with an element of order 36; the second is the only one with an element of order 9 and more than one element of order 2; and the third is the only one with an element of order 4 and more than two elements of order 3.

None of the second four are isomorphic because the third is the only one with an element of order 9; the first is the only one which has an element of order 6 which commutes with every other element of the group; and the last is the only one for which no element other than the identity commutes with every other element of the group. One could also complete the argument by counting elements of order 2 in the three remaining groups.

(2) Let G be a subgroup of S_6 , and assume that G has an element of order 6. Show that G contains a normal subgroup of index 2.

Solution: we claim that $G \cap A_6$ is a subgroup of index 2 in G . It is clear that the intersection of any two subgroups is a subgroup. Since it remains true in G that a product of odd permutations is even, and an odd times an even is odd, we see that as long as G contains an odd permutation, $G \cap A_6$ has index 2. Since G contains an element of order 6, it suffices to show that every element of S_6 of order 6 is odd. Let $\sigma \in S_6$ have order 6. Considering the representation of σ as a product of disjoint cycles, we see that there are two possibilities: either σ is a cycle of order 6, or it is a product of two disjoint cycles of orders 2 and 3. In both cases, we can write it as a product of 7 transpositions, so σ must be odd. Therefore $G \cap A_6$ is a subgroup of index 2, and we conclude it must be normal.

(3) How many distinct isomorphisms are there from S_4 to itself?

Solution: This problem was substantially harder than intended. The answer is 24, and we show this by bounding it from above and below by 24.

To obtain the upper bound, we consider $\sigma = (1, 2, 3, 4)$ and $\tau = (3, 2, 1)$ in S_4 . We note that $\sigma\tau = (1, 4)$, so we have the relations $\sigma^4 = \tau^3 = \sigma\tau\sigma\tau = e$. We claim that every element of S_4 can be written as a product of σ 's and τ 's, i.e. that σ and τ generate S_4 . Since every element of S_4 can be written as a product of transpositions, it is enough to show that every transposition can be obtained as a product of σ 's and τ 's. We have at least $(1, 4) = \sigma\tau$. Noting that $\sigma^{-1} = \sigma^3$ and $\tau^{-1} = \tau^2$, using the earlier homework problem we see that $\sigma(1, 4)\sigma^{-1} = (2, 1)$, $\sigma(2, 1)\sigma^{-1} = (3, 2)$, and $\sigma(3, 2)\sigma^{-1} = (4, 3)$. This gives 4 of the 6 transpositions in S_4 . We obtain the last two as $\tau^2(1, 4)\tau^{-2} = (2, 4)$ and $\sigma(2, 4)\sigma^{-1} = (3, 1)$. Therefore, every element of S_4 is a product of σ 's and τ 's, and by definition of a homomorphism, it follows that every homomorphism of S_4 to another group G is determined by where σ and τ are sent to.

Now, suppose that f is an isomorphism from S_4 to itself. There are 6 elements of order 4 in S_4 , and $f(\sigma)$ must be one of these; say $f(\sigma) = (a, b, c, d)$ with a, b, c, d distinct in $\{1, 2, 3, 4\}$. Similarly, $f(\tau)$ must be an element of order 3, say $f(\tau) = (e, f, g)$. Let us consider the case that d is the number not appearing in e, f, g (the other cases being the same); then by reordering (e, f, g) we can set $e = a$, so that $f(\tau) = (a, f, g)$ with either $f = b, g = c$, or $f = c, g = b$. We see that in the first case, we get $f(\sigma)f(\tau) = (a, c, b, d)$, and in the second case, we get $f(\sigma)f(\tau) = (d, a)$. Because f is a homomorphism, we must have $e = f(e) = f(\sigma\tau\sigma\tau) = f(\sigma)f(\tau)f(\sigma)f(\tau)$, so we must have that $f(\sigma)f(\tau)$ has order 2. This means that the only possibility above was $f(\tau) = (a, c, b)$. However, this depended on specifying that d did not appear in τ ; we could also have had a, b , or c not appear in τ , so we find 4 possibilities for $f(\tau)$ given our choice of $f(\sigma)$, and we see that there are at most $6 \cdot 4 = 24$ possibilities for f .

To establish the lower bound, we consider a different approach: given any $\sigma \in S_4$, we define a function $f_\sigma : S_4 \rightarrow S_4$ by sending τ to $\sigma\tau\sigma^{-1}$. We first claim that for any $\sigma \in S_4$, in fact f_σ is an isomorphism. It is clearly a homomorphism, since $f_\sigma(\tau_1\tau_2) = \sigma\tau_1\tau_2\sigma^{-1} = \sigma\tau_1\sigma^{-1}\sigma\tau_2\sigma^{-1} = f_\sigma(\tau_1)f_\sigma(\tau_2)$. Since S_4 is finite, to show that f_σ is bijective, it suffices to show it is injective. But suppose that $f_\sigma(\tau_1) = f_\sigma(\tau_2)$; i.e., $\sigma\tau_1\sigma^{-1} = \sigma\tau_2\sigma^{-1}$. Then we can use the cancellation law to conclude $\tau_1 = \tau_2$. Therefore, f_σ is an isomorphism (and so far we have used no special properties of S_4).

We finally claim that for any distinct $\sigma_1, \sigma_2 \in S_4$, we have that f_{σ_1} is different from f_{σ_2} . We see this as follows: we can consider f as a function $f : S_4 \rightarrow \text{Sym}(S_4)$, since each f_σ is bijective. In fact, this is a homomorphism, since $f_{\sigma\tau} = f_\sigma \circ f_\tau$. So, to check injectivity, it is enough to show that only the trivial permutation in S_4 maps to the identity, which is to say that if we have a σ such that $\sigma\tau\sigma^{-1} = \tau$ for all $\tau \in S_4$, then $\sigma = e$. Another way to put this is that if $\sigma\tau = \tau\sigma$ for all τ , then $\sigma = e$, and this is easy to check directly. Therefore, we have constructed an injective map from S_4 into the set of isomorphisms of S_4 to itself, and since $|S_4| = 24$, we find there are

at least 24 isomorphisms from S_4 to itself. Combining this with our upper bound from before, we get the desired statement.

(4) Let G be a finite group, and let S be the set of elements $g \in G$ such that $g = g^{-1}$. Show that $|G|$ is even if and only if S has more than one element.

Solution: we first observe that $g = g^{-1}$ if and only if $g^2 = e$, so we either have either $g = e$, or $o(g) = 2$. In particular, if S has more than one element, G has an element of order 2, and by Lagrange's theorem, must have even order. For the converse, we consider the set T of elements of G not in S . T consists of the elements for which $g \neq g^{-1}$, which means that the elements of T can be paired up naturally by pairing g with g^{-1} . Thus, T always has an even number of elements. If $|G|$ is even, then S must also have an even number of elements, and since $e \in S$, S cannot have 0 elements, and must have at least 2.