

## Solutions

**Exercise 1.** Suppose  $G$  has two subgroups  $H, K$  with  $K \triangleleft G$ . Let  $HK = \{hk : h \in H, k \in K\}$ . Prove that  $HK$  is a subgroup of  $G$ .

*Proof.* Let  $h_1k_1$  and  $h_2k_2$  be two elements of  $HK$ . Then

$$h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1h_2^{-1}h_2k_1k_2^{-1}h_2^{-1}.$$

We have  $h_1h_2^{-1} \in H$  and since  $K \triangleleft G$  we have  $h_2k_1k_2^{-1}h_2^{-1} \in K$ . Hence  $h_1k_1(h_2k_2)^{-1} \in HK$ , so  $HK$  is a subgroup of  $G$ . □

**Exercise 2.** For each of the groups  $G = S_3, A_4, S_4$ , find subgroups  $H, K$  with  $K \triangleleft G$ , such that  $G = HK$ . Prove your claims.

*Solution.*

$G$	$H$	$K$
$S_3$	$\langle(12)\rangle$	$\langle(123)\rangle$
$A_4$	$\langle(123)\rangle$	$K_4$
$S_4$	$S_3$	$K_4$
$S_4$	$\langle(12)\rangle$	$A_4$

(There are two possible answers for  $S_4$ .) □

**Exercise 3.** This exercise shows how to recognize direct products. Let  $G$  be a group with normal subgroups  $H \triangleleft G$  and  $K \triangleleft G$ . Assume that  $H \cap K = \{e\}$  and  $HK = G$ . Prove that  $G \simeq H \times K$ .

*Hint:* First show that  $hkh^{-1}k^{-1} = e$  for all  $h \in H, k \in K$ . Then show that the function  $f : H \times K \rightarrow G$  given by  $f(h, k) = hk$  is a group isomorphism.

*Proof.* For all  $h \in H, k \in K$ , we have

$$hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} = h(kh^{-1}k^{-1}) \in H \cap K,$$

so  $hkh^{-1}k^{-1} = e$ . This means that  $hk = kh$ .

Now prove that  $f$  is an isomorphism, as follows. For  $h, h' \in H$  and  $k, k' \in K$ , we compute:

$$f(h, k) \cdot f(h', k') = (hk)(h'k') = h(kh')k' = h(h'k)k' = (hh')(kk') = f(hh', kk') = f((h, k) \cdot (h', k')).$$

Hence  $f$  is a homomorphism. Since  $HK = G$ , any element  $g \in G$  may be written as  $g = hk$  for some  $h \in H, k \in K$ , and  $f(h, k) = hk = g$ , so  $f$  is surjective. Finally, if  $(h, k) \in \ker f$ , then  $hk = e$ , so  $h = k^{-1} \in H \cap K = \{e\}$ , so  $(h, k) = (e, e)$ . Hence  $f$  is injective. □

**Exercise 4.** Let  $G$  be a nonabelian group of order  $2n$ , where  $n \geq 3$ . Suppose there exist elements  $a, b \in G$  such that  $a$  has order  $n$ ,  $b$  has order 2, and  $bab^{-1} = a^{-1}$ . Prove that  $G \simeq D_n$ .

*Hint:* First show that  $G = \{a^i b^j : 0 \leq i < n, 0 \leq j \leq 1\}$ , then find elements in  $D_n$  analogous to  $a, b$  and use all this to define an isomorphism  $f : G \rightarrow D_n$ .

*Proof.* The subgroup  $\langle a \rangle$  has index two in  $G$ . It does not contain  $b$ , lest  $a = bab^{-1} = a^{-1}$ , contradicting  $n \geq 3$ . Hence

$$G = \langle a \rangle \cup \langle a \rangle b = \{a^i b^j : 0 \leq i < n, 0 \leq j \leq 1\}.$$

In  $D_n$ , let  $r$  be a rotation by  $2\pi/n$  and let  $s$  be a reflection. Then  $r$  has order  $n$  and  $s$  has order 2. Using a picture, you can check that  $srs^{-1} = r^{-1}$ . Hence we also have

$$D_n = \{r^i s^j : 0 \leq i < n, 0 \leq j \leq 1\}.$$

Define  $f : G \rightarrow D_n$  by  $f(a^i b^j) = r^i s^j$ . We must show that it is a group homomorphism. Now

$$a^i b^j a^k b^\ell = \begin{cases} a^{i+k} b^\ell & \text{if } j = 0 \\ a^{i-k} b^{\ell+1} & \text{if } j = 1. \end{cases}$$

So

$$\begin{aligned} f(a^i b^j a^k b^\ell) &= \begin{cases} r^{i+k} s^\ell & \text{if } j = 0 \\ r^{i-k} s^{\ell+1} & \text{if } j = 1 \end{cases} \\ &= r^i s^j r^k s^\ell \\ &= f(a^i b^j) f(a^k b^\ell). \end{aligned}$$

Therefore  $f$  is a group homomorphism, as claimed. It is clear that  $f$  is a bijection, so it is an isomorphism.  $\square$

**Exercise 5.** Suppose that  $G$  is a nonabelian group of order  $2p$ , where  $p > 2$  is a prime. Prove that  $G \simeq D_p$ .

Hint: Find elements  $a, b$  as in the previous exercise.

*Proof.* Since  $G$  has even order, it contains an element  $b$  of order 2. (Proved in exam 1 study problems.) Since  $G$  is nonabelian, it cannot have all elements of order 2 (proved in hw), nor can it have an element of order  $2p$ , lest it be cyclic. Hence  $G$  has an element  $a$  of order  $p$ . The subgroup  $\langle a \rangle$  has index two in  $G$ , hence is normal, so  $bab^{-1} = a^k$  for some  $k$ . Conjugating again by  $b$ , we have

$$a = b^2 ab^{-2} = (a^k)^k = a^{k^2}.$$

hence  $p \mid k^2 - 1$ . Since  $p$  is prime, we have  $p \mid k - 1$  or  $p \mid k + 1$ . But if  $p \mid k - 1$  then  $bab^{-1} = a^k = a$  so  $G$  would be abelian, which it is not. Hence  $p \mid k + 1$ , so  $bab^{-1} = a^{-1}$ . By the result in the previous problem, we have  $G \simeq D_{2p}$ .  $\square$

**Exercise 6.** Let  $G$  be a finite group, let  $a \in G$ , and let  $A$  be the conjugacy class of  $a$  in  $G$ , and let  $H = C_G(a)$  be the centralizer of  $a$  in  $G$ . Prove that

$$|A| = \frac{|G|}{|H|}.$$

Hint: Show that the function  $f : G/H \rightarrow A$  given by  $f(gH) = gag^{-1}$  is a well-defined bijection.

*Proof.* We first show that  $f$  is well-defined. Suppose  $gH = kH$ . Then  $g = kh$  for some  $h \in H$ . We have  $f(gH) = gag^{-1} = khah^{-1}k^{-1}$ . Since  $h \in H = C_G(a)$ , we have  $hah^{-1} = a$ . Therefore  $f(gH) = kak^{-1} = f(kH)$ , so  $f$  is well-defined. By definition,  $A = \{gag^{-1} : g \in G\}$ . So if  $b \in A$  there is  $g \in G$  such that  $b = gag^{-1}$ , and  $f(gH) = gag^{-1} = b$ . Hence  $f$  is surjective. Finally, suppose  $f(gH) = f(kH)$ , for some  $g, k \in G$ . Then  $gag^{-1} = kak^{-1}$ , so  $k^{-1}gag^{-1}k^{-1} = a$ . This means  $k^{-1}g \in H$ , so  $kH = gH$ . Hence  $f$  is injective. Since  $f : G/H \rightarrow A$  is a bijection, we have  $|A| = |G/H| = |G|/|H|$ .  $\square$

Comment: Note that  $H$  is not necessarily normal in  $G$ . The set  $G/H$  is not a group, and  $f$  is not a group homomorphism. That's why we could not prove injectivity by showing  $\ker f$  is trivial.

**Exercise 7.** Use the first isomorphism theorem to prove that  $S_4/K_4 \simeq S_3$ .

*Proof.* It suffices to find a surjective homomorphism  $f : S_4 \rightarrow S_3$  with  $\ker f = K_4$ . We have found such a homomorphism in Exercise 5 of hw 3 (see also Exercise 6 of hw 3). By the first isomorphism theorem, it follows that the map

$$\bar{f} : S_4/K_4 \rightarrow S_3$$

given by  $\bar{f}(\sigma K_4) = f(\sigma)$ , for  $\sigma \in S_4$ , is an isomorphism.  $\square$