Solutions

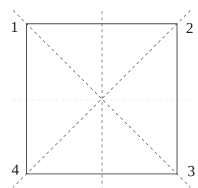
Exercise 1. On a group G define a relation by: $a \sim b$ if there exists $g \in G$ such that $b = gag^{-1}$. Prove that this is an equivalence relation.

Proof. Let $a, b, c \in G$.
Since $a = eae^{-1}$, we have $a \sim a$ so the relation is *reflexive*.
If $a \sim b$, then $\exists g \in G$ such that $gag^{-1} = b$. Then $a = g^{-1}bg$, so $b \sim a$. Hence the relation is *symmetric*.
If $a \sim b$ and $b \sim c$, there are $g, h \in G$ such that $b = gag^{-1}$ and $c = hbh^{-1}$. Then $c = hga(hg)^{-1}$ so $a \sim c$. Hence the relation is *transitive*. □

Exercise 2. The equivalence classes under the equivalence relation of exercise 1 are called *conjugacy classes*. Find the conjugacy classes in S_3, D_4 and A_4.

Solution.
The conjugacy classes in S_3 are
$\{e\}$
$\{(12), (13), (23)\}$, (2-cycles)
$\{(123), (321)\}$ (3-cycles).

For D_4, we number the square as in hw 2:



The congugacy classes in D_4 are
$\{e\}$
$\{(13)(24)\}$ (180 degree rotation),
$\{(14)(23), (12)(34)\}$ (edge reflections),
$\{(13), (24))\}$ (vertex reflections),
$\{(1234), (4321)\}$ (90 degree rotations).

Alternative solution with matrices:
$\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$
$\{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\}$ (180 degree rotation),
$\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$ (edge reflections),
$\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\}$ (vertex reflections),
$\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\}$ (90 degree rotations).

The conjugacy classes in A_4 are
$\{e\}$,
$\{(12)(34), (13)(24), (23)(14)\}$, (edge reflections of tetrahedron),
$\{(123), (134), (142), (243))\}$ (face rotations in same direction),
$\{(123), (134), (142), (243))\}$ (face rotations in the other direction).

□

Exercise 3. For $G = S_3$, choose one element a in each conjugacy class and compute the order of its centralizer $C_G(a)$. Do the same for $G = D_4$ and $G = A_4$. What relation do you observe between the order $C_G(a)$ and the number of elements in the conjugacy class of a?

Solution.

a	$C_{S_3}(a)$	$\|C_{S_3}(a)\|$	$\|$conj. class of $a\|$
e	S_3	6	1
(12)	$\langle(12)\rangle$	2	3
(123)	$\langle(123)\rangle$	3	2

a	$C_{D_4}(a)$	$\|C_{D_4}(a)\|$	$\|$conj. class of $a\|$
e	D_4	8	1
(13)(24)	D_4	8	1
(12)(34)	$\langle(12),(34)\rangle$	4	2
(13)	$\langle(13),(24)\rangle$	4	2
(1234)	$\langle(1234)\rangle$	4	2

a	$C_{A_4}(a)$	$\|C_{A_4}(a)\|$	$\|$conj. class of $a\|$
e	A_4	12	1
(12)(34)	$\langle(12)(34),(13)(24)\rangle$	4	3
(123)	$\langle(123)\rangle$	3	4
(321)	$\langle(123)\rangle$	3	4

In all cases we have the relation

$$|C_G(a)| \cdot |\text{conj. class of } a| = |G|.$$

(This holds for any element a of any finite group G, as we will prove later.) □

Exercise 4. Let $(a_1, a_2, \ldots, a_k)$ be a cycle of length k in S_n, and let $\sigma \in S_n$ be an arbitrary permutation. Prove that

$$\sigma(a_1, a_2, \ldots, a_k)\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)).$$

Proof. Let $b_i = \sigma(a_i)$, for $1 \le i \le k$. Let $c \in \{1, 2, \ldots, n\}$. If $c \notin \{b_1, \ldots, b_k\}$, then $\sigma^{-1}c \notin \{a_1, \ldots, a_k\}$, so

$$\sigma(a_1, a_2, \ldots, a_k)\sigma^{-1}(c) = \sigma\sigma^{-1}(c) = c = (b_1, \ldots, b_k)(c).$$

If $c = b_i$, then $\sigma^{-1}(c) = a_i$, so reading subscripts modulo k, we have

$$\sigma(a_1, a_2, \ldots, a_k)\sigma^{-1}(c) = \sigma(a_1, a_2, \ldots, a_k)a_i = \sigma(a_{i+1}) = b_{i+1} = (b_1, \ldots, b_k)(c).$$

Hence the permutions $\sigma(a_1, a_2, \ldots, a_k)\sigma^{-1}$ and $(b_1, \ldots, b_k)$ have the same effect on every number in $\{1, 2, \ldots, n\}$, so they are the same permutation. □

Exercise 5. The subset $\{(12)(34), (13)(24), (14)(23)\} \subset S_4$ is a conjugacy-class. Number the elements as $x_1 = (12)(34)$, $x_2 = (13)(24)$, $x_3 = (14)(23)$. If $\sigma \in S_4$ is an arbitrary permutation, then $\sigma x_1 \sigma^{-1} = x_i$, for some $i \in \{1, 2, 3\}$. Likewise, $\sigma x_2 \sigma^{-1} = x_j$ and $\sigma x_3 \sigma^{-1} = x_k$, for some $j, k \in \{1, 2, 3\}$. Thus, we have a permutation

$$f(\sigma) = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}.$$

Prove that this defines a homomorphism $f : S_4 \to S_3$ and compute $\ker f$ and $\operatorname{im} f$.

Proof. To see that f is a homomorphism, we take two elements $\sigma, \tau \in S_4$ and compute:

$$f(\sigma\tau)(x_i) = \sigma\tau x_i(\sigma\tau)^{-1} = \sigma\tau x_i \tau^{-1}\sigma^{-1},$$

and

$$f(\sigma)f(\tau)(x_i) = f(\sigma)(\tau x_i \tau^{-1}) = \sigma\tau x_i \tau^{-1}\sigma^{-1},$$

so $f(\sigma\tau) = f(\sigma)f(\tau)$.

The kernel of f consists of those $\sigma \in S_4$ such that $\sigma x_i \sigma^{-1} = x_i$ for all i. In other words,

$$\ker f = C_{S_4}(x_1) \cap C_{S_4}(x_2) \cap C_{S_4}(x_3).$$

Each centralizer is isomorphic to D_4. In exercise 1 we listed the elements of $C_{S_4}(x_2)$. Of these, those which commute with x_1 are the 22-cycles and e. These also commute with x_3, so we have

$$\ker f = \{e, (12)(34), (13)(24), (14)(23)\}.$$

The image of f is all of S_3. To see this, it suffices to find σ, τ in S_4 such that $f(\sigma) = (12)$ and $f(\tau) = (23)$. Since

$$(23)x_1(23) = x_2, \quad (23)x_3(23) = x_3, \quad (34)x_2(34) = x_3, \quad (34)x_1(34) = x_1,$$

it follows that $f((23)) = (12)$ and $f((34)) = (23)$, so $\operatorname{im} f = S_3$, as claimed. □

Exercise 6. Let C be the group of rigid motions of a cube. Judson [Thm. 4.12] defines an isomorphism $g : C \to S_4$, where $g(\sigma)$ is the permutation of $\{1, 2, 3, 4\}$ induced by the action of σ on the four diagonals of the cube. From the previous exercise, there must also be a homomorphism $f : C \to S_3$. Give a geometric construction of f by finding three things in the cube permuted by C.

Solution. The homomorphism $f : C \to S_3$ is given by permuting the three perpendicular lines through opposite faces of the cube. Each line is the axis of a 180 degree rotation, which corresponds to a 22-cycle in S_4. If L is one of these lines and $x \in C$ is 180 degree rotation about L, then any $\sigma \in C$ sends L to the line σL whose 180 degree rotation is $\sigma x \sigma^{-1}$. So σ permutes the lines in the same way it permutes the 22-cycles under conjugation. Hence this is the same homomorphism as in the previous exercise. □