Solutions

Exercise 1. Let $G = \langle a \rangle$ be a finite cyclic group of order n, with generator a. Prove that a^k generates G if and only if $\gcd(k, n) = 1$.

Proof. ($\Rightarrow$) Suppose that a^k generates G. Then a is some power of a^k, so we have $a = (a^k)^\ell = a^{kl}$ for some integer ℓ. This means that $kl \equiv 1 \mod n$, so that $[k]_n$ is a unit in $\mathbb{Z}_n$. We have proved that this implies that $\gcd(k, n) = 1$.

($\Leftarrow$) Suppose that $\gcd(k, n) = 1$. Then there are integers x, y such that $kx + ny = 1$. We compute

$$a = a^1 = a^{kx+ny} = (a^k)^x (a^n)^y = (a^k)^x,$$

since $a^n = 1$. If $g \in G$ is an arbitrary element, there is an integer i such that $g = a^i$, and we have

$$g = a^i = (a^k)^{xi} \in \langle a^k \rangle.$$

Hence a^k generates G. □

Exercise 2. Suppose that $n = p^k$, where p is a prime and k is a positive integer. Find the number of generators of G.

Solution. From the previous problem, a^d is *not* a generator of G iff $p \mid d$. The set $\{d \in \mathbb{Z} : \ 1 \leq d \leq p^k, \ p \mid d\} = \{p, 2p, \ldots, p^{k-1}p\}$ has p^{k-1} elements. This is the number of non-generators of G. Hence G has $p^k - p^{k-1} = p^{k-1}(p-1)$ generators. □

Exercise 3. Use Lagrange's theorem to prove that any group of prime order is cyclic.

Proof. Let G be a group whose order is a prime p. Since $p > 1$, there is an element $a \in G$ such that $a \neq e$. The group $\langle a \rangle$ generated by a is a subgroup of G. By Lagrange's theorem, the order of $\langle a \rangle$ divides $|G|$. But the only divisors of $|G| = p$ are 1 and p. Since $a \neq e$ we have $|\langle a \rangle| > 1$, so $|\langle a \rangle| = p$. Hence $\langle a \rangle = G$ and G is cyclic. □

Exercise 4. Let G be a group and let $a \in G$. Show that the subset $C_G(a) = \{b \in G : ba = ab\}$, consisting of the elements of G which commute with a, is a subgroup of G.

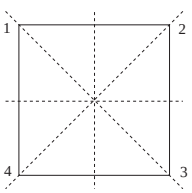
Proof. The identity element e commutes with every element of G, so $e \in C_G(a)$. If $x \in C_G(a)$ then $xa = ax$. Multiplying both sides of this equation on the left by x^{-1}, we get $a = x^{-1}ax$. Multiplying both sides of this new equation on the right by x^{-1}, we get $ax^{-1} = x^{-1}a$. Hence $x^{-1} \in C_G(a)$. Let $x, y \in C_G(a)$. Then x and y both commute with a, so we have

$$xya = xay = axy,$$

hence $xy \in C_G(a)$. Hence $C_G(a)$ is a subgroup of G.

□

Exercise 5. The dihedral group D_4 is the symmetry group of a square. Number the vertices of a square as shown:



Each element of D_4 gives a permutation of the vertices, which is an element of the group S_4. Let $H \subset S_4$ be the set of permutations arising from symmetries of the square. Let $\sigma = (13)(24)$ be the permutation switching $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$. Prove that $H = C_{S_4}(\sigma)$ (see Exercise 4).

Proof. First check that every element of D_4 commutes with the 180° rotation σ. It follows that every permutation coming from D_4 commutes with $(13)(24)$. So we have a chain of subgroups

$$D_4 < C_{S_4}(\sigma) < S_4.$$

Since $|D_4| = 8$ and $|S_4| = 24$, Lagrange's theorem implies that $|C_{S_4}(\sigma)|$ is a number divisible by 8 and dividing 24. The only possible orders are $8, 24$, so we have either $C_{S_4}(\sigma) = D_4$ or $C_{S_4}(\sigma) = S_4$. Since there are permutations that do not commute with σ, eg (12), it follows that $C_{S_4}(\sigma) \neq S_4$, so we must have $C_{S_4}(\sigma) = D_4$. □

Exercise 6. Let T be the group of rigid motions of a regular tetrahedron. Show that $|T| = 12$ (Hint: imitate the proof of Prop. 4.11 in Judson) and list the orders of all the elements in T.

Proof. Every symmetry sends a face to one of four faces in three possible ways, so $|T| = 12$. There are 8 rotations of order three, two for each axis through a vertex and the midpoint of the opposite face. There are 3 rotations of order 2, about axes through pairs of opposite edges. With the identity element of order 1, we have $8 + 3 + 1 = 12$ elements of T, which is all of T, since $|T| = 12$. □