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Solutions

Exercise 1. Let $G = \{e, x, y\}$ be any group with three elements. Without knowing the group law, fill in the Cayley table.

Solution. The product xy must be one of e, x, y. If $xy = x$ then $y = e$, which it is not. Likewise, $xy \neq y$. Hence $xy = e$ and $y = x^{-1}$. Now $x^2 \neq x$ (lest $x = e$) and $x^2 \neq e$, (lest $x = x^{-1} = y$) so $x^2 = y$. Likewise $yx = e$ and $y^2 = x$. Hence G has multiplication table

$\circ$	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

□

Exercise 2. Let $G = \{e, x, y, z\}$ be a group with four elements. Again, you are not told the group law. Show that there are exactly two possibilities for the Cayley table.

Solution. If $x^2 = y^2 = z^2 = e$ then the product of any two of these is the third, and we get the table on the left. Otherwise, some element, say x, does not square to e. Then x^2 is either y or z, say y. Now xy is either e or z. but if $xy = e$ then $xz \neq e$ (lest $z = x^{-1} = y$) and $xz \neq y$ (lest $z = x$), and $xz \neq x, z$ as before. So we cannot have $xy = e$, so $xy = z$. We now have $x^2 = y$, $x^3 = z$, so $x^4 = e$. We see that G is cyclic, generated by x, and get the table on the right. If you take a different element to be one not squaring to e, then you get the same table, but with the rows and columns permuted.

$\circ$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

$\circ$	e	x	y	z
e	e	x	y	z
x	x	y	z	e
y	y	z	e	x
z	z	e	x	y

□

Exercise 3. Let G be a group and let $g_1, g_2, \ldots, g_n$ be elements of G. Prove that

$$(g_1 g_2 \cdots g_n)^{-1} = g_n^{-1} g_{n-1}^{-1} \cdots g_2^{-1} g_1^{-1}.$$

Proof. It is obvious for $n = 1$. For $n = 2$, we have

$$(g_2^{-1} \cdot g_1^{-1})(g_1 \cdot g_2) = g_2^{-1} \cdot e \cdot g_2 = g_2^{-1} g_2 = e,$$

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and likewise $(g_1 \cdot g_2)(g_2^{-1} \cdot g_1^{-1}) = e$. Suppose now that $n \geq 2$. Let $g = g_1 g_2 \cdots g_{n-1}$. By induction, we have

$$g^{-1} = g_{n-1}^{-1} \cdots g_1^{-1}.$$

From the case $n = 2$, we have

$$(g_1 g_2 \cdots g_n)^{-1} = (g \cdot g_n)^{-1} = g_n^{-1} g^{-1} = g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1}.$$

□

Exercise 4. Let $\mathbb{Z}_n^\times$ be the group of units of $\mathbb{Z}_n$ and assume that $n \geq 3$. Prove that there is an element $a \in \mathbb{Z}_n^\times$ such that $a^2 = 1$, but $a \neq 1$.

Proof. Taking $a = [-1]$ does the job: we have $[-1]^2 = [(-1)^2] = [1]$, and $-1 \not\equiv 1 \mod n$ since $n \geq 3$. □

Exercise 5. Let G be a group for which $g^2 = e$ for all $g \in G$. Prove that G is abelian.

Proof. Let $x, y \in G$. We have $x^2 = y^2 = (xy)^2 = e$. Multiplying both sides of the equation

$$e = (xy)(xy)$$

on the left by x and on the right by y gives

$$xy = x(xy)(xy)y = (x^2)(yx)(y^2) = e(yx)e = yx.$$

Hence $xy = yx$ for all $x, y \in G$ so G is abelian. □

Exercise 6. Let G be the symmetry group of an equilateral triangle, and let $a, b \in G$ be two reflections. Write the remaining three non-identity elements of G in terms of a and b.

Solution. We have $G = \{e,\ a,\ b,\ aba,\ ab,\ ba\}$. The third reflection is $aba = bab$ and the two rotations of order three are ab, ba. □