AbstractAlgebra1

You will have **50 minutes** to complete this exam. The total score is **100 points**. To receive full credit, your solution must be **correct**, **complete**, and **legible**. Justify your conclusions.

Notation:

$\mathbb{R}$ is the set of real numbers.
$\mathbb{Z}^+$ is the set of positive integers.
$\iota_{\mathbb{R}}$ is the identity function on $\mathbb{R}$.

---

1. (10+10 points)

(a) State the definition of a **function** mapping a set A into a set B.

A **function** mapping A to B is a subset f of $A \times B$ satisfying the following condition: for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$.

(b) State the definition of a **partition** of a set S.

A **partition** of a set S is a set $\mathcal{P}$ of subsets of S such that each element of S is a member of exactly one set in $\mathcal{P}$.

2. (10+10+10+10 points)
TRUE or FALSE? Justify your answers.

(a) $\{\emptyset, \{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$.

FALSE. $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ but $\{\emptyset\} \notin \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$, because $\{\emptyset\} \neq \emptyset$ and $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$.

(b) The function $h\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ defined so that for each positive integer n, $h(n)$ is the number of decimal digits of n, is neither one-to-one, nor onto.

FALSE. h is onto, because for every $k \in \mathbb{Z}^+$ the number $n = 10^k - 1 = 99\ldots9$ (with k 9's) is in $\mathbb{Z}^+$ and satisfies $h(n) = k$.

(c) If $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(0) = 1$ and $f(x) = x$ for all nonzero $x \in \mathbb{R}$, then there exists a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g \neq \iota_{\mathbb{R}}$ and $f \circ g = f$.

TRUE. Let $g = f$. Then $f \neq \iota_{\mathbb{R}}$, because $f(0) = 1 \neq 0 = \iota_{\mathbb{R}}(0)$, and $f \circ f = f$, because for all $x \in \mathbb{R}$, $f(x) \neq 0$, hence $f(f(x)) = f(x)$.

(d) If a binary structure $(S, *)$ has elements a and b such that $a * b \neq b * a$, then the operation $*$ is not commutative.

TRUE. If $a * b \neq b * a$ for some elements $a, b \in S$, then the property '$a * b = b * a$ for all $a, b \in S$' fails, so $*$ is not commutative.

3. (10+10 points)

Let $\sim$ be the relation on the set $\mathbb{R} \times \mathbb{R}$ defined by

$$(a_1, a_2) \sim (b_1, b_2) \quad \text{if and only if} \quad a_1 + b_2 = a_2 + b_1.$$

(a) Prove that $\sim$ is an equivalence relation.

- $\sim$ is reflexive: For arbitrary pair $(a_1, a_2) \in \mathbb{R} \times \mathbb{R}$ we have $a_1 + a_2 = a_2 + a_1$, proving $(a_1, a_2) \sim (a_1, a_2)$.
- $\sim$ is symmetric: If $(a_1, a_2) \sim (b_1, b_2)$, that is, $a_1 + b_2 = a_2 + b_1$, then also $b_1 + a_2 = b_2 + a_1$, hence $(b_1, b_2) \sim (a_1, a_2)$.
- $\sim$ is transitive: If $(a_1, a_2) \sim (b_1, b_2)$ and $(b_1, b_2) \sim (c_1, c_2)$, that is, $a_1 + b_2 = a_2 + b_1$ and $b_1 + c_2 = b_2 + c_1$, then $(a_1 + b_2) + (b_1 + c_2) = (a_2 + b_1) + (b_2 + c_1)$, so subtracting $b_1 + b_2$ from both sides yields that $a_1 + c_2 = a_2 + c_1$. Hence $(a_1, a_2) \sim (c_1, c_2)$.

(b) Visualizing the elements of $\mathbb{R} \times \mathbb{R}$ as points in the plane, give a geometric description of the equivalence class containing a given point (a_1, a_2).

The equivalence class of (a_1, a_2) is

$$\begin{aligned} \overline{(a_1, a_2)} &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid a_1 + y = a_2 + x\} \\ &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x + (a_2 - a_1)\}, \end{aligned}$$

which is the set of points of the line with slope 1 passing through the point (a_1, a_2).

4. (20 points)
Define an operation $*$ on $\mathbb{Z}$ by $x * y = x + y - xy$ for all $x, y \in \mathbb{Z}$. Prove that the function $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(x) = 1 - x$ for all $x \in \mathbb{Z}$ is an isomorphism of $(\mathbb{Z}, \cdot)$ with $(\mathbb{Z}, *)$.

- φ is one-to-one, because if $m, n \in \mathbb{Z}$ are such that $\varphi(m) = \varphi(n)$, that is, $1 - m = 1 - n$, then $m = n$.
- φ is onto, because for each $k \in \mathbb{Z}$, $k = 1 - (1 - k) = \varphi(1 - k)$ for $1 - k \in \mathbb{Z}$.
- φ has the homomorphism property: for arbitrary $x, y \in \mathbb{Z}$,

$$\begin{aligned} \varphi(xy) &= 1 - xy, \\ \varphi(x) * \varphi(y) &= (1 - x) * (1 - y) \\ &= (1 - x) + (1 - y) - (1 - x)(1 - y) \\ &= 1 - xy, \end{aligned}$$

hence $\varphi(xy) = \varphi(x) * \varphi(y)$.