Second Midterm

All questions are worth ten points. In addition to the questions, there will be an additional ten points to be awarded for style and clarity in writing.

1) Definitions

1. Define the **index** $|G : H|$ of a subgroup $H < G$.
2. Define the **external direct product** $G_1 \oplus G_2$ of groups G_1 and G_2. (Specify the set of elements that make up the direct product and its group operation.)

2) Fill in the blanks or answer True/False.

1. $11^{29} \equiv$ __ (mod 29).
2. True or False: Every group G of order 7 contains an element $a \in G$ such that $|a| = 7$. __
3. True or False: if $\mathrm{Aut}(G_1) \approx \mathrm{Aut}(G_2)$ then $G_1 \approx G_2$. __
4. In S_5, $|(12)(13)(456)| =$ __. (Compose left to right.)
5. True or False: Let G be an arbitrary finite group of order n. If $d|n$, then there is an $H < G$ of order d. __

3) Lagrange's Theorem

1. Give a clear and complete statement of Lagrange's Theorem for a subgroup H of a finite group G.

2. Use Lagrange's Theorem to prove that the order of an element $a \in G$ divides the order of G.

3. Use Lagrange's Theorem to prove that if $|G|$ is prime, then G must be cyclic.

4. If H and K are subgroups of G, where $|H| = 33$ and $|K| = 28$, what are the possible orders of $H \cap K$? Explain your answer.

4) List the distinct left cosets of H in G if

1. $G = D_4$, $H = \langle FR \rangle$, where R is a rotation and F is a flip.

2. $G = \mathbb{Z}/15\mathbb{Z}$, $H = \langle 3 \rangle$.

5) Suppose $\phi : G_1 \rightarrow G_2$ is an isomorphism. Prove that G_1 is abelian if and only if G_2 is abelian.

6) Cosets: Given a subgroup $H < G$, prove that any two cosets of H have the same order, that is, for any $a, b \in G$, $|aH| = |bH|$.

7) Classification: Prove the following:
Let G be a group of order $2p$, for $p > 2$ prime. Suppose G is not cyclic and that $a \in G$ has order p. If $b \notin \langle a \rangle$, show that $|b| = 2$.

8) Let $|x| = 40$. List all the elements of $\langle x \rangle$ that have order 10. Explain your answer.

9) (extra credit) Prove Lagrange's Theorem. You may cite basic properties of cosets, such as those listed in Gallian's Lemma, if you state them accurately.