Abstract algebra Exam
Name:

25 points per question.
The best four questions count.

Question 1

Let $f(x) = x^4 + 5x - 6$ and $g(x) = x^3 + 3x^2 - 4$.

- Find, with proof, the greatest common denominator (f, g) of f and g and write (f, g) in the form $af + bg$, for suitable polynomials a and b.
- Hence write f and g as products of irreducible polynomials in the rings $\mathbb{Q}[x]$, $\mathbb{R}[x]$, $\mathbb{C}[x]$ and $\mathbb{Z}_7[x]$.

Question 2

Find all solutions of the following polynomial equations and hence factor the polynomials completely; if a polynomial is irreducible, say why.

- $f(x) = x^2 + x + 3 = 0 \mod 11$.

- $g(x) = x^3 + x + 4 = 0 \mod 17$.

- $h(x) = x^4 - 8x^2 - 20 = 0$ in $\mathbb{R}[x]$ and in $\mathbb{C}[x]$.

Question 3

Let $p(x) = x^4 + 3x^2 + 4$.
Write, with proof, $p(x)$ as a product of irreducibles in:

- $\mathbb{Z}_7[x]$
- $\mathbb{Z}[x]$
- $\mathbb{R}[x]$
- $\mathbb{C}[x]$
- In the field with four elements, $\mathbb{Z}_2[t]/(t^2 + t + 1)$.

Question 4

Let $\mathbb{S} = \mathbb{Z}_3[x]/(x^2+1)$.

- Prove that $\mathbb{S}$ is a field.
- How many elements of $\mathbb{S}$ are there?
  Explain your answer and list the distinct elements.
- Write out the multiplication table for $\mathbb{S}$ and identify all those elements of $\mathbb{S}$ that are perfect squares.
- Using your multiplication table and the quadratic formula, or otherwise, find all solutions of the equation $z^2 - z - 1 = 0$ in the field $\mathbb{S}$ and hence factorize the polynomial $z^2 - z - 1$ completely as a product of irreducible polynomials with coefficients in $\mathbb{S}$.

Question 5

Find all ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_6$ and for each determine its image and kernel.
How many ring homomorphisms are there from $\mathbb{Z}_6$ to $\mathbb{Z}_{12}$?
Explain your answer.

Question 6

Let $\mathbb{A} = \mathbb{Z}_6$, $\mathbb{B} = \mathbb{Z}_8$, $\mathbb{C} = \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{D} = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{E} = \mathbb{Z}_2[x]/(x^3 + x + 1)$. Decide with proof, which, if any, of the rings $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{E}$ are pairwise isomorphic.