Abstract algebra Exam
Name:
Twenty points per question
The best five questions will count.

Question 1

Let $a = 6409$ and $b = 3536$.

Using the Euclidean algorithm, find the greatest common divisor (a, b) of a and b and express (a, b) as a linear combination of a and b with integer coefficients.

Use your results to factor a and b as products of primes.

Question 2

For each of the following equations, determine, with proof, if it is solvable and for each solvable equation find all integer solutions; if an equation is not solvable prove that it is not solvable:

- $4x = 3 \mod 13$.

- $16x = 1 \mod 23$.

- $40x = 81 \mod 15$.

Question 3

For each of the following polynomial equations, determine with proof if it is solvable and for each solvable equation find all integer solutions; if an equation is not solvable prove that it is not solvable: also use your results to factor the given polynomial, if possible.

- $f(x) = x^2 + x + 1 = 0 \mod 5$.

- $g(x) = x^2 + 5x + 3 = 0 \mod 13$.

- $h(x) = x^3 - 1 = 0 \mod 7$.

Question 4

Given positive integers a, b, c and d, put $(a, b) = p$ and $(c, d) = q$.
Prove that pq divides (ac, bd).
Give an example to show that (ac, bd) can be equal to pq and another example to show that (ac, bd) can be more than pq.

Question 5

- Find all integer solutions x of the following system:

$$\begin{aligned} 37x &= 7 \mod 13 \\ 9x &= 5 \mod 11 \end{aligned}$$

- Also find the number of integer solutions x of the system with $|x| < 1000$.

Question 6

Consider the following addition and multiplication tables for a ring:

$+$	x	z	y	w
x	z	x	w	y
z	x	z	y	w
y	w	y	x	z
w	y	w	z	x

$\cdot$	x	z	y	w
x	z	z	x	x
z	z	z	z	z
y	x	z	y	w
w	x	z	w	y

Show that this ring is really the ring $\mathbb{Z}_4$ in disguise.
Hint: begin by identifying the additive and multiplicative identities.

Question 7

For x in $\mathbb{Z}$ put $f(x) = 5x \mod 20$.
Prove that the map f is a ring homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_{20}$.
Also determine the image of the map f and its kernel.

Question 8

Let $\mathbb{A} = \mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{B} = \mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{C} = \mathbb{Z}_{12}$.
Decide, with proof, which of the rings $\mathbb{A}$, $\mathbb{B}$, or $\mathbb{C}$ are isomorphic to each other.