Final Exam

Answer ten of the following twelve problems. If more than ten are answered, only the best ten count.

Warning: Some questions are copies of previous test questions, but others are modifications of previous test questions or new questions. Be sure to read every question carefully.

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1. Define ring.

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2. Define group.

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3. Given an equivalence relation ~, prove that $a \sim b$ if and only if $[a] = [b]$

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4. A ring R is called a Boolean ring if $a^2 = a$ for every element a of R.
If R is a Boolean ring and $a, b \in R$, prove that $(a + b)^2 = a^2 + b^2$ and that $(a + b)^3 = a^3 + b^3$ and that, in fact, for all integers $n > 0$, $(a + b)^n = a^n + b^n$.

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5. Use the Euclidean algorithm to write the greatest common divisor of 382 and 26 as a linear combination of 382 and 26.

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6. If $\theta : R \to S$ is a homomorphism of the ring R into the ring S and $a \in R$, prove that

$$\theta(a^n) = \theta(a)^n \quad \text{for every positive integer } n.$$

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7. Prove that $\mathbf{Z}_n$ is an integral domain if and only if n is prime.

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8. Factor the following polynomials into irreducible factors over the field $\mathbf{Z}_7$. Write the elements of $\mathbf{Z}_7$ as 0, 1, 2, 3, 4, 5, 6. [Hints: They both factor over $\mathbf{Z}_7$ into different factors than over the real numbers. Recall that a polynomial of degree two or three factors if and only if it has a root.]

$$f(x) = x^2 + 3$$

$$g(x) = x^3 + 2x^2 + 5x + 3$$

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9. Prove: If K is a nonempty subset of a group G so that for all $a, b \in K$, $ab^{-1} \in K$, then K is a subgroup of G.

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10. Prove that if a commutative ring R has a nonzero divisor of zero then R cannot be an ordered ring.

11. An element a of a ring R is said to be idempotent if $a^2 = a$. If m and n are relatively prime integers greater than 1, prove that the ring $\mathbf{Z}_{mn}$ has at least two idempotent elements other than 0 and 1. [Hint: If $1 = mx + ny$, consider mx and ny as elements of $\mathbf{Z}_{mn}$.]

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12. Given a homomorphism θ from a group G to a group H, we define K, the kernel of θ, to be the set of all elements of G that map to the identity in H.

That is, $K = \{ a \in G : \theta(a) = e_H \}$

Show (a) that K is a subgroup of G and (b) that K also satisfies the property:

$\forall k \in K \; \forall a \in G \; (a^{-1} k a \in K)$

That is, for all k in K and all a in G, the product $a^{-1} k a$ is in K.
